

Tight Bounds on the Approximability of Almost-satisfiable Horn SAT

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The only three non-trivial Boolean CSPs for which satisfiability is polynomial time decidable. [Schaefer'78]

- LIN-mod-2 – linear equations modulo 2
- 2-SAT
- Horn-SAT – a CNF formula where each clause consists of at most one unnegated literal
 - x_1, \bar{x}_2
 - $\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4$
 - $x_2 \wedge x_4 \rightarrow x_5$ (equivalent to $\bar{x}_2 \vee \bar{x}_4 \vee x_5$)

Robust algorithms for almost satisfiable instances I

A small ϵ fraction of constraints of a satisfiable instance were corrupted by noise. Can we still find a good assignment?

Finding almost satisfying assignments

Given an instance which is $(1 - \epsilon)$ -satisfiable, can we efficiently find an assignment satisfying $(1 - f(\epsilon) - o(1))$ constraints, where $f(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$?

Robust algorithms for almost satisfiable instances II

No for $(1 - \epsilon)$ -satisfiable LIN-mod-2.

Yes for $(1 - \epsilon)$ -satisfiable 2-SAT.

Yes for $(1 - \epsilon)$ -satisfiable Horn-SAT

Robust algorithms for almost satisfiable instances II

No for $(1 - \epsilon)$ -satisfiable LIN-mod-2.

- NP-Hard to find a $(1/2 + \epsilon)$ -satisfying solution. [Håstad'01]

Yes for $(1 - \epsilon)$ -satisfiable 2-SAT.

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Yes for $(1 - \epsilon)$ -satisfiable 2-SAT.

- SDP based algorithm finds a $(1 - O(\sqrt{\epsilon}))$ -satisfying assignment. [CMM'09]
- Tight under Unique Games Conjecture. [KKMO'07]

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Yes for $(1 - \epsilon)$ -satisfiable Horn-SAT

- LP based algorithm finds a $(1 - O(\frac{\log \log(1/\epsilon)}{\log(1/\epsilon)}))$ -satisfying assignment. [Zwick'98]
- For Horn-3SAT, Zwick's algorithm gives a $(1 - \frac{1}{\log(1/\epsilon)})$ -satisfying solution, losing an exponentially large factor.
- Is it tight?

Bounds on approximability of almost satisfiable Horn-SAT

Previously known

	Horn-3SAT	Horn-2SAT
Approx. Alg.	$1 - O\left(\frac{1}{\log(1/\epsilon)}\right)$ [Zwick'98]	$1 - 3\epsilon$ [KSTW'00]
NP-Hardness	$1 - \epsilon^c$ for some $c < 1$ [KSTW'00]	$1 - 1.36\epsilon$ from Vertex Cover
UG-Hardness		$1 - (2 - \delta)\epsilon$ from Vertex Cover

Bounds on approximability of almost satisfiable Horn-SAT

Our result

	Horn-3SAT	Horn-2SAT
Approx. Alg.	$1 - O\left(\frac{1}{\log(1/\epsilon)}\right)$ [Zwick'98]	$1 - 2\epsilon$
NP-Hardness	$1 - \epsilon^c$ for some $c < 1$ [KSTW'00]	$1 - 1.36\epsilon$ from Vertex Cover
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Bounds on approximability of almost satisfiable Horn-SAT

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Why rely on UGC? Isn't there a subexponential time algorithm [ABS'10] for UGC ?

- Even for $(1 - \epsilon)$ -satisfiable 2-SAT, the NP-hardness of finding $(1 - \omega_\epsilon(1)\epsilon)$ -satisfying assignment is not known without assuming UGC, – while UGC implies the optimal $(1 - \Omega(\sqrt{\epsilon}))$ hardness.
- People also trying to prove UGC these days...
[Khot-Moshkovitz'10]

Part I.

Theorem

Given a $(1 - \epsilon)$ -satisfiable instance for Horn-2SAT, it is possible to find a $(1 - 2\epsilon)$ -satisfying assignment efficiently.

Part II.

Theorem

There exists absolute constant $C > 0$, s.t. for every $\epsilon > 0$, given a $(1 - \epsilon)$ -satisfiable instance for Horn-3SAT, it is UG-hard to find a $(1 - \frac{C}{\log(1/\epsilon)})$ -satisfying assignment.

Theorem

Given a $(1 - \epsilon)$ -satisfiable instance for Horn-2SAT, it is possible to find a $(1 - 2\epsilon)$ -satisfying assignment efficiently.

▶ Go to Part II...

Warm up – approximation preserving reduction from Vertex Cover to Horn-2SAT

Given a Vertex Cover instance $G = (V, E)$,

- Each variable x_i in Horn-2SAT corresponds a vertex $v_i \in V$.
- For each $e = (v_i, v_j) \in E$, introduce a clause $\bar{x}_i \vee \bar{x}_j$ of weight $\frac{1}{|E|+1}$.
- For each $v_i \in V$, introduce a clause x_i of weight $\frac{1}{(|E|+1)|V|}$.

Observation,

- Exists optimal solution violating no edge clause.
- For this optimal solution, set of violated vertex clauses \sim set of vertices chosen in optimal Vertex Cover solution.

Therefore, $1 - OPT(\text{Horn2SAT}) = OPT(\text{Vertex Cover}) / (|E| + 1)$.

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Robust algorithm for almost-satisfiable Horn-2SAT

In Min Horn-2SAT Deletion problem, the goal is to find a subset of clauses of minimum total weight whose deletion makes the instance satisfiable.

We prove

Theorem

There is a polynomial-time 2-approximation algorithm for Min Horn-2SAT Deletion problem.

This directly implies

Theorem

Given a $(1 - \epsilon)$ -satisfiable instance for Horn-2SAT, it is possible to find a $(1 - 2\epsilon)$ -satisfying assignment efficiently.

Approximation algorithm for Min Horn-2SAT Deletion

Possible clauses in Horn-2SAT

- “True constraint”: x_i
- “False constraint”: \bar{x}_i
- “Disjunction constraint”: $\bar{x}_i \vee \bar{x}_j$
- “Implication constraint”: $x_i \rightarrow x_j$ (equivalent to $\bar{x}_i \vee x_j$)

LP Formulation as follows, we have $\text{OPT}_{\text{LP}} \leq \text{OPT}$.

$$\begin{array}{ll} \text{min.} & \sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i + \sum_{i < j} w_{ij}^{(D)} z_{ij}^{(D)} + \sum_{i \neq j} w_{ij}^{(I)} z_{ij}^{(I)} \\ \text{s.t.} & z_{ij}^{(D)} \geq y_i + y_j - 1 \quad \forall i < j \\ & z_{ij}^{(I)} \geq y_i - y_j \quad \forall i \neq j \\ & z_{ij}^{(D)} \geq 0 \quad \forall i < j \\ & z_{ij}^{(I)} \geq 0 \quad \forall i \neq j \\ & y_i \in [0, 1] \quad \forall i \in V \end{array}$$

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$$\begin{aligned} \text{min.} \quad & \sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i + \sum_{i < j} w_{ij}^{(D)} z_{ij}^{(D)} + \sum_{i \neq j} w_{ij}^{(I)} z_{ij}^{(I)} \\ \text{s.t.} \quad & z_{ij}^{(D)} \geq \max\{y_i + y_j - 1, 0\} \quad \forall i < j \\ & z_{ij}^{(I)} \geq \max\{y_i - y_j, 0\} \quad \forall i \neq j \\ & y_i \in [0, 1] \quad \forall i \in V \end{aligned}$$

Approximation algorithm for Min Horn-2SAT Deletion

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Approximation algorithm for Min Horn-2SAT Deletion

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LP Formulation as follows, we have $\text{OPT}_{\text{LP}} \leq \text{OPT}$.

$$\begin{aligned} \min. \quad \text{Val}(f) &= \sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)}y_i \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i + y_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i - y_j, 0\} \\ \text{s.t.} \quad f = \{y_i\} &\in [0, 1]^V \end{aligned}$$

Half-integrality and rounding I

Lemma

Given a solution $f = \{y_i\}$, we can efficiently convert f into $f^ = \{y_i^*\}$ such that each $y_i^* \in \{0, 1, 1/2\}$ is half-integral, and $\text{Val}(f^*) \leq \text{Val}(f)$.*

Corollary

We can efficiently find an optimal LP solution and all the variables in the solution are half-integral.

Rounding

Given an optimal LP solution $f = \{y_i\}$ which is half-integral, define $f_{\text{int}} = \{x_i\}$ as follows.

For each $i \in V$, let $x_i = 0$ when $y_i \leq 1/2$, and $x_i = 1$ when $y_i = 1$.

Half-integrality and rounding II

Observation

- $x_i \leq y_i$ and $1 - x_i \leq 2(1 - y_i)$.
- $\max\{x_i + x_j - 1, 0\} \leq \max\{y_i + y_j - 1, 0\}$ (by $x_i \leq y_i, x_j \leq y_j$).
- $\max\{x_i - x_j, 0\} \leq 2 \max\{y_i - y_j, 0\}$.
 - When $y_i \leq y_j$,
 $x_i \leq x_j \Rightarrow \max\{x_i - x_j, 0\} = \max\{y_i - y_j, 0\} = 0$.
 - When $y_i > y_j \Rightarrow y_i - y_j \geq 1/2$,
 $\max\{x_i - x_j, 0\} \leq 1 \leq 2 \max\{y_i - y_j, 0\}$.

Half-integrality and rounding II

Observation

- $x_i \leq y_i$ and $1 - x_i \leq 2(1 - y_i)$.
- $\max\{x_i + x_j - 1, 0\} \leq \max\{y_i + y_j - 1, 0\}$ (by $x_i \leq y_i, x_j \leq y_j$).
- $\max\{x_i - x_j, 0\} \leq 2 \max\{y_i - y_j, 0\}$.

Therefore,

$$\begin{aligned}\text{Val}(f_{\text{int}}) &= \sum_{i \in V} w_i^{(T)}(1 - x_i) + \sum_{i \in V} w_i^{(F)}x_i \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{x_i + x_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{x_i - x_j, 0\} \\ &\leq \sum_{i \in V} w_i^{(T)}2(1 - y_i) + \sum_{i \in V} w_i^{(F)}y_i \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i + y_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)}2 \max\{y_i - y_j, 0\} \\ &\leq 2\text{Val}(f) = 2\text{OPT}_{\text{LP}} \leq 2\text{OPT}.\end{aligned}$$

Proof of half-integrality lemma I

Given $f = \{y_i\}$, construct pairs of critical points

$$W_f = \{(p, 1 - p) : 0 \leq p \leq 1/2, \exists i \in V, \text{s.t. } p = y_i \vee 1 - p = y_i\}.$$

Idea. Iteratively revise f , so that W_f contains less “non-half-integral” pairs after each iteration, while not increasing $\text{Val}(f)$. Done when W_f contains no “non-half-integral” pair.

Proof of half-integrality lemma II

$$W_f = \{(p, 1 - p) : 0 \leq p \leq 1/2, \exists i \in V, \text{s.t. } p = y_i \vee 1 - p = y_i\}$$

In each iteration. Choose a non-half-integral pair $(p, 1 - p) \in W_f$ ($0 < p < 1/2$). Let

$$S = \{i : y_i = p\}, S' = \{i : y_i = 1 - p\}.$$

Let a and b be the two “neighbors” of p in W_f . I.e., let

$$a = \max\{q < p : (q, 1 - q) \in W_f, 0\},$$
$$b = \min\{q > p : (q, 1 - q) \in W_f, 1/2\}.$$

Define

$$f^{(t)} = \{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}.$$

Claim

$\text{Val}(f^{(t)})$ is linear with $t \in [a, b]$.

Exists $\tau \in \{a, b\}$ such that $\text{Val}(f^{(\tau)}) \leq \text{Val}(f^{(p)}) = \text{Val}(f)$. Update f by $f^{(\tau)}$, we have one less non-half-integral pair $(p, 1 - p)$ in W_f .

Proof of Claim

$$\begin{aligned} \text{Val}(f^{(t)}) &= \sum_{i \in V} w_i^{(T)} (1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)} \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\} \\ f^{(t)} &= \frac{\{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}}{\quad} \end{aligned}$$

Only need to prove $g_1(t) = \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\}$ and

$g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any i, j .

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- $i, j \in V \setminus (S \cup S')$. g_1 and g_2 are constant functions.

Proof of Claim

$$\begin{aligned} \text{Val}(f^{(t)}) &= \sum_{i \in V} w_i^{(T)} (1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)} \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\} \\ f^{(t)} &= \frac{\{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}}{} \end{aligned}$$

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$g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any i, j .

- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$,
 - The only “non-linear point” is $t = 1 - y_i$ for g_1 and $t = y_i$ for g_2 – they are away from $[a, b]$.

Proof of Claim

$$\begin{aligned} \text{Val}(f^{(t)}) &= \sum_{i \in V} w_i^{(T)} (1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)} \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\} \\ f^{(t)} &= \frac{\{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}}{\quad} \end{aligned}$$

Only need to prove $g_1(t) = \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\}$ and

$g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any i, j .

- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$, or $i \in S \cup S', j \in V \setminus (S \cup S')$. ✓
- $i \in S, j \in S'$ (or $i \in S', j \in S$).
 - $g_1(t) = y_i^{(t)} + y_j^{(t)} - 1 \equiv 0$ is constant function.
 - When $i \in S, j \in S', y_i^{(t)} \leq y_j^{(t)}$, $g_2(t) \equiv 0$ is constant function.
 - When $i \in S', j \in S, y_i^{(t)} \geq y_j^{(t)}$, $g_2(t) = y_i^{(t)} - y_j^{(t)}$ is linear function of t .

Proof of Claim

$$\begin{aligned} \text{Val}(f^{(t)}) &= \sum_{i \in V} w_i^{(T)} (1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)} \\ &\quad + \sum_{i < j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\} \\ f^{(t)} &= \frac{\{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}}{} \end{aligned}$$

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- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$, or $i \in S \cup S', j \in V \setminus (S \cup S')$. ✓
- $i \in S, j \in S'$ (or $i \in S', j \in S$). ✓
- $i, j \in S$ (or $i, j \in S'$).
 - When $i, j \in S$, $y_i^{(t)} + y_j^{(t)} < 1$, $g_1(t) \equiv 0$ is constant function.
 - When $i, j \in S'$, $y_i^{(t)} + y_j^{(t)} > 1$, $g_1(t) = y_i^{(t)} + y_j^{(t)} - 1$ is linear function of t .
 - $y_i^t = y_j^t$, thus $g_2(t) \equiv 0$ is constant function.

Proof of Claim

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- $i \in V \setminus (S \cup S'), j \in S \cup S'$, or $i \in S \cup S', j \in V \setminus (S \cup S')$. ✓
- $i \in S, j \in S'$ (or $i \in S', j \in S$). ✓
- $i, j \in S$ (or $i, j \in S'$). ✓

Q.E.D.

Theorem

There exists absolute constant $C > 0$, s.t. for every $\epsilon > 0$, given a $(1 - \epsilon)$ -satisfiable instance for Horn-3SAT, it is UG-hard to find a $(1 - \frac{C}{\log(1/\epsilon)})$ -satisfying assignment.

Theorem [Raghavendra'08]

*There is a canonical SDP relaxation $\text{SDP}(\Lambda)$ for each CSP Λ .
Let $1 > c > s > 0$. A c vs. s integrality gap instance for $\text{SDP}(\Lambda)$
 \Rightarrow UG-hardness of $(c - \eta)$ vs. $(s + \eta)$ gap- Λ problem, for every
constant $\eta > 0$.*

We prove the UG-hardness by showing

Theorem

*There is a $(1 - 2^{-\Omega(k)})$ vs. $(1 - 1/k)$ gap instance for
 $\text{SDP}(\text{Horn-3SAT})$, for every $k > 1$.*

The canonical SDP for Boolean CSPs I

\mathcal{C} : The set of constraints over $X = \{x_1, x_2, \dots, x_n \in \{0, 1\}\}$.

For each $C \in \mathcal{C}$, set up a local distribution π_C on all truth-assignments $\{\sigma : X_C \rightarrow \{0, 1\}\}$.

- Introduce scalar variables $\pi_C(\sigma)$ with non-negativity constraints and $\sum_{\sigma} \pi_C(\sigma) = 1$.

A lifted LP (in Sherali-Adams system).

$$\begin{aligned} \text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}} [\mathbf{Pr}_{\sigma \in \pi_C} [C(\sigma) = 1]] \\ \text{s.t.} \quad & \mathbf{Pr}_{\sigma \in \pi_C} [\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = X_{(x_i, b_1), (x_j, b_2)} \\ & \forall C \in \mathcal{C}, x_i, x_j \in C, b_1, b_2 \in \{0, 1\} \end{aligned}$$

An example I

Instance. Clause 1 : $x_1 \wedge x_2 \rightarrow x_4$, Clause 2: $x_3 \wedge x_4 \rightarrow x_2$.

Objective. Maximize

$$\frac{1}{2}(\pi_1(x_1, x_2, x_4) + \pi_1(\bar{x}_1, x_2, x_4) + \pi_1(x_1, \bar{x}_2, x_4) + \pi_1(\bar{x}_1, \bar{x}_2, x_4) + \pi_1(\bar{x}_1, x_2, \bar{x}_4) + \pi_1(x_1, \bar{x}_2, \bar{x}_4) + \pi_1(\bar{x}_1, \bar{x}_2, \bar{x}_4)) + \frac{1}{2}(\pi_2(x_3, x_4, x_2) + \pi_2(\bar{x}_3, x_4, x_2) + \pi_2(x_3, \bar{x}_4, x_2) + \pi_2(\bar{x}_3, \bar{x}_4, x_2) + \pi_2(\bar{x}_3, x_4, \bar{x}_2) + \pi_2(x_3, \bar{x}_4, \bar{x}_2))$$

Constraints.

$\pi_1(\cdot, \cdot, \cdot)$ and $\pi_2(\cdot, \cdot, \cdot)$ form distributions respectively.

$$\pi_1(x_1, x_2, x_4) + \pi_1(\bar{x}_1, x_2, x_4) = \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \bar{x}_3, x_4) = X_{(x_2,1),(x_4,1)}$$

$$\pi_1(x_1, \bar{x}_2, x_4) + \pi_1(\bar{x}_1, \bar{x}_2, x_4) = \pi_2(\bar{x}_2, x_3, x_4) + \pi_2(\bar{x}_2, \bar{x}_3, x_4) = X_{(x_2,0),(x_4,1)}$$

$$\pi_1(x_1, x_2, \bar{x}_4) + \pi_1(\bar{x}_1, x_2, \bar{x}_4) = \pi_2(x_2, x_3, \bar{x}_4) + \pi_2(x_2, \bar{x}_3, \bar{x}_4) = X_{(x_2,1),(x_4,0)}$$

An example II

$$\begin{aligned} & \pi_1(x_1, \bar{x}_2, \bar{x}_4) + \pi_1(\bar{x}_1, \bar{x}_2, \bar{x}_4) = \pi_2(\bar{x}_2, x_3, \bar{x}_4) + \pi_2(\bar{x}_2, \bar{x}_3, \bar{x}_4) = \\ & X_{(x_2,0),(x_4,0)} \\ & \pi_1(x_1, x_2, x_4) + \pi_1(\bar{x}_1, x_2, x_4) + \pi_1(x_1, x_2, \bar{x}_4) + \pi_1(\bar{x}_1, x_2, \bar{x}_4) = \\ & \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \bar{x}_3, x_4) + \pi_2(x_2, x_3, \bar{x}_4) + \pi_2(x_2, \bar{x}_3, \bar{x}_4) = \\ & X_{(x_2,1),(x_2,1)} \\ & \pi_1(x_1, \bar{x}_2, x_4) + \pi_1(\bar{x}_1, \bar{x}_2, x_4) + \pi_1(x_1, \bar{x}_2, \bar{x}_4) + \pi_1(\bar{x}_1, \bar{x}_2, \bar{x}_4) = \\ & \pi_2(\bar{x}_2, x_3, x_4) + \pi_2(\bar{x}_2, \bar{x}_3, x_4) + \pi_2(\bar{x}_2, x_3, \bar{x}_4) + \pi_2(\bar{x}_2, \bar{x}_3, \bar{x}_4) = \\ & X_{(x_2,0),(x_2,0)} \\ & \pi_1(x_1, x_2, x_4) + \pi_1(\bar{x}_1, x_2, x_4) + \pi_1(x_1, \bar{x}_2, x_4) + \pi_1(\bar{x}_1, \bar{x}_2, x_4) = \\ & \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \bar{x}_3, x_4) + \pi_2(\bar{x}_2, x_3, x_4) + \pi_2(\bar{x}_2, \bar{x}_3, x_4) = \\ & X_{(x_4,1),(x_4,1)} \\ & \pi_1(x_1, x_2, \bar{x}_4) + \pi_1(\bar{x}_1, x_2, \bar{x}_4) + \pi_1(x_1, \bar{x}_2, \bar{x}_4) + \pi_1(\bar{x}_1, \bar{x}_2, \bar{x}_4) = \\ & \pi_2(x_2, x_3, \bar{x}_4) + \pi_2(x_2, \bar{x}_3, \bar{x}_4) + \pi_2(\bar{x}_2, x_3, \bar{x}_4) + \pi_2(\bar{x}_2, \bar{x}_3, \bar{x}_4) = \\ & X_{(x_4,0),(x_4,0)} \end{aligned}$$

The canonical SDP for Boolean CSPs II

Add vectors. Introduce $\mathbf{v}_{(x,0)}$ and $\mathbf{v}_{(x,1)}$ corresponding to the events $x = 0$ and $x = 1$.

Constraints.

- $\mathbf{v}_{(x,0)} \cdot \mathbf{v}_{(x,1)} = 0$ – mutually exclusive events
- $\mathbf{v}_{(x,0)} + \mathbf{v}_{(x,1)} = \mathbf{1}$ – probability adds up to 1
- $\Pr_{\sigma \in \pi_C} [\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = \mathbf{v}_{(x_i,b_1)} \cdot \mathbf{v}_{(x_j,b_2)}$

The canonical SDP.

$$\text{max.} \quad \mathbf{E}_{C \in \mathcal{C}} [\Pr_{\sigma \in \pi_C} [C(\sigma) = 1]]$$

$$\text{s.t.} \quad \mathbf{v}_{(x_i,0)} \cdot \mathbf{v}_{(x_i,1)} = 0$$

$$\mathbf{v}_{(x_i,0)} + \mathbf{v}_{(x_i,1)} = \mathbf{1} \quad \forall i \in [n]$$

$$\|\mathbf{1}\|^2 = 1 \quad \forall i \in [n]$$

$$\Pr_{\sigma \in \pi_C} [\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = \mathbf{v}_{(x_i,b_1)} \cdot \mathbf{v}_{(x_j,b_2)} \\ \forall C \in \mathcal{C}, x_i, x_j \in C, b_1, b_2 \in \{0, 1\}$$

The canonical SDP for Boolean CSPs III

Simplification. Define $\mathbf{v}_{(x,1)} = \mathbf{v}_x$, and $\mathbf{v}_{(x,0)} = \mathbf{1} - \mathbf{v}_x$. The canonical SDP is equivalent to

$$\begin{aligned} \text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}} [\Pr_{\sigma \in \pi_C} [C(\sigma) = 1]] \\ \text{s.t.} \quad & (\mathbf{1} - \mathbf{v}_{x_i}) \cdot \mathbf{v}_{x_j} = 0 \quad \forall i \in [n] \\ & \|\mathbf{1}\|^2 = 1 \quad \forall i \in [n] \\ & \Pr_{\sigma \in \pi_C} [\sigma(x_i) = 1 \wedge \sigma(x_j) = 1] = \mathbf{v}_{x_i} \cdot \mathbf{v}_{x_j} \quad \forall C \in \mathcal{C}, x_i, x_j \in C \end{aligned}$$

The canonical SDP for Boolean CSPs III

Simplification. Define $\mathbf{v}_{(x,1)} = \mathbf{v}_x$, and $\mathbf{v}_{(x,0)} = \mathbf{I} - \mathbf{v}_x$. The canonical SDP is equivalent to

$$\begin{aligned} \text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}} [\Pr_{\sigma \in \pi_C} [C(\sigma) = 1]] \\ \text{s.t.} \quad & (\mathbf{I} - \mathbf{v}_{x_i}) \cdot \mathbf{v}_{x_i} = 0 \quad \forall i \in [n] \\ & \|\mathbf{I}\|^2 = 1 \quad \forall i \in [n] \\ & \Pr_{\sigma \in \pi_C} [\sigma(x_i) = 1 \wedge \sigma(x_j) = 1] = \mathbf{v}_{x_i} \cdot \mathbf{v}_{x_j} \quad \forall C \in \mathcal{C}, x_i, x_j \in C \end{aligned}$$

Comment

The SDP is stronger than lifted LP in many cases. For 2-SAT, lifted LP has a huge gap 1 vs. 3/4, while SDP gives the optimal gap $(1 - \epsilon)$ vs. $(1 - O(\sqrt{\epsilon}))$.

Gap instance

Consider instance $\mathcal{I}_k^{\text{Horn}}$.

$$\begin{array}{ll} \text{Step 0:} & x_0, y_0 \\ \text{Step 1:} & x_0 \wedge y_0 \rightarrow x_1, x_0 \wedge y_0 \rightarrow y_1 \\ \text{Step 2:} & x_1 \wedge y_1 \rightarrow x_2, x_1 \wedge y_1 \rightarrow y_2 \\ \text{Step 3:} & x_2 \wedge y_2 \rightarrow x_3, x_2 \wedge y_2 \rightarrow y_3 \\ & \dots \\ \text{Step } k+1: & x_k \wedge y_k \rightarrow x_{k+1}, x_k \wedge y_k \rightarrow y_{k+1} \\ \text{Step } k+2: & \overline{x_{k+1}}, \overline{y_{k+1}} \end{array}$$

Gap instance

Consider instance $\mathcal{I}_k^{\text{Horn}}$.

Step 0:

$$x_0, y_0$$

Step 1:

$$x_0 \wedge y_0 \rightarrow x_1, \quad x_0 \wedge y_0 \rightarrow y_1$$

Step 2:

$$x_1 \wedge y_1 \rightarrow x_2, \quad x_1 \wedge y_1 \rightarrow y_2$$

Step 3:

$$x_2 \wedge y_2 \rightarrow x_3, \quad x_2 \wedge y_2 \rightarrow y_3$$

...

Step $k + 1$:

$$x_k \wedge y_k \rightarrow x_{k+1}, \quad x_k \wedge y_k \rightarrow y_{k+1}$$

Step $k + 2$:

$$\overline{x_{k+1}}, \quad \overline{y_{k+1}}$$

Observation

$\mathcal{I}_k^{\text{Horn}}$ is not satisfiable. Therefore $\text{OPT}(\mathcal{I}_k^{\text{Horn}}) \leq 1 - \Omega(1/k)$.

A good solution for lifted LP

$$\begin{array}{ll} \text{Step 0:} & x_0, y_0 \\ \text{Step 1:} & x_0 \wedge y_0 \rightarrow x_1, x_0 \wedge y_0 \rightarrow y_1 \\ \text{Step 2:} & x_1 \wedge y_1 \rightarrow x_2, x_1 \wedge y_1 \rightarrow y_2 \\ \text{Step 3:} & x_2 \wedge y_2 \rightarrow x_3, x_2 \wedge y_2 \rightarrow y_3 \\ & \dots \\ \text{Step } k+1: & x_k \wedge y_k \rightarrow x_{k+1}, x_k \wedge y_k \rightarrow y_{k+1} \\ \text{Step } k+2: & \overline{x_{k+1}}, \overline{y_{k+1}} \end{array}$$

Observation

Clauses from two different steps share at most one variable. No need to worry about pairwise margins.

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

$$\text{loss} = 2\delta$$

$x_0(y_0)$	$\pi_C(\sigma)$	\Rightarrow	$x_0 \wedge y_0 \rightarrow x_1(y_1)$	$\pi_C(\sigma)$
1	$1 - \delta$		$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
0	δ		$0 \wedge 1 \rightarrow 0$	δ
			$1 \wedge 0 \rightarrow 0$	δ

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

$$\text{loss} = 2\delta$$

$x_1 \wedge y_1 \rightarrow x_2(y_2)$	$\pi_C(\sigma)$		$x_0 \wedge y_0 \rightarrow x_1(y_1)$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 4\delta$	\Leftarrow	$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	2δ		$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	2δ		$1 \wedge 0 \rightarrow 0$	δ

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

$$\text{loss} = 2\delta$$

$x_1 \wedge y_1 \rightarrow x_2(y_2)$	$\pi_C(\sigma)$		$x_2 \wedge y_2 \rightarrow x_3(y_3)$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 4\delta$	\Rightarrow	$1 \wedge 1 \rightarrow 1$	$1 - 8\delta$
$0 \wedge 1 \rightarrow 0$	2δ		$0 \wedge 1 \rightarrow 0$	4δ
$1 \wedge 0 \rightarrow 0$	2δ		$1 \wedge 0 \rightarrow 0$	4δ

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

$$\text{loss} = 2\delta$$

$x_k \wedge y_k \rightarrow x_{k+1}$	$\pi_C(\sigma)$	$\Leftarrow \dots$	$x_2 \wedge y_2 \rightarrow x_3(y_3)$	$\pi_C(\sigma)$
$x_k \wedge y_k \rightarrow y_{k+1}$			$1 \wedge 1 \rightarrow 1$	
$1 \wedge 1 \rightarrow 1$	$1 - 2^{k+1}\delta$		$0 \wedge 1 \rightarrow 0$	4δ
$0 \wedge 1 \rightarrow 0$	$2^k\delta$		$1 \wedge 0 \rightarrow 0$	4δ
$1 \wedge 0 \rightarrow 0$	$2^k\delta$			

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

$$\text{loss} = 2\delta + 2(1 - 2^{k+1}\delta)$$

$x_k \wedge y_k \rightarrow x_{k+1}$		$\pi_C(\sigma)$	\Rightarrow	$x_{k+1}(y_{k+1})$		$\pi_C(\sigma)$
$x_k \wedge y_k \rightarrow y_{k+1}$				1		$1 - 2^{k+1}\delta$
$1 \wedge 1 \rightarrow 1$		$1 - 2^{k+1}\delta$		1		$1 - 2^{k+1}\delta$
$0 \wedge 1 \rightarrow 0$		$2^k\delta$		0		$2^{k+1}\delta$
$1 \wedge 0 \rightarrow 0$		$2^k\delta$				

A good solution for lifted LP

Step 0:	x_0 ,	y_0
Step 1:	$x_0 \wedge y_0 \rightarrow x_1$,	$x_0 \wedge y_0 \rightarrow y_1$
Step 2:	$x_1 \wedge y_1 \rightarrow x_2$,	$x_1 \wedge y_1 \rightarrow y_2$
Step 3:	$x_2 \wedge y_2 \rightarrow x_3$,	$x_2 \wedge y_2 \rightarrow y_3$
	...	
Step $k + 1$:	$x_k \wedge y_k \rightarrow x_{k+1}$,	$x_k \wedge y_k \rightarrow y_{k+1}$
Step $k + 2$:	$\overline{x_{k+1}}$,	$\overline{y_{k+1}}$

loss = $2\delta + 2(1 - 2^{k+1}\delta) = 1/2^k$ (by choosing $\delta = 1/2^{k+1}$).

$x_k \wedge y_k \rightarrow x_{k+1}$		$\pi_C(\sigma)$	\Rightarrow	$x_{k+1}(y_{k+1})$		$\pi_C(\sigma)$
$x_k \wedge y_k \rightarrow y_{k+1}$						
$1 \wedge 1 \rightarrow 1$		$1 - 2^{k+1}\delta$		1		$1 - 2^{k+1}\delta$
$0 \wedge 1 \rightarrow 0$		$2^k \delta$		0		$2^{k+1} \delta$
$1 \wedge 0 \rightarrow 0$		$2^k \delta$				

Add vectors to get a good SDP solution

Problem. Is there any set of vectors corresponding to the following distribution?

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

Add vectors to get a good SDP solution

Problem. Is there any set of vectors corresponding to the following distribution?

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

The required inner-product matrix – should be PSD.

	1	\mathbf{v}_{x_i}	\mathbf{v}_{y_i}	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$
1	1	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{x_i}	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{y_i}	$1 - \delta$	$1 - 2\delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
$\mathbf{v}_{x_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	
$\mathbf{v}_{y_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$		$1 - 2\delta$

Add vectors to get a good SDP solution

Problem. Is there any set of vectors corresponding to the following distribution?

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

The required inner-product matrix – should be PSD.

	$\mathbf{1}$	\mathbf{v}_{x_i}	\mathbf{v}_{y_i}	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$
$\mathbf{1}$	1	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{x_i}	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{y_i}	$1 - \delta$	$1 - 2\delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
$\mathbf{v}_{x_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$?
$\mathbf{v}_{y_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$?	$1 - 2\delta$

Add vectors to get a good SDP solution

Problem. Is there any set of vectors corresponding to the following distribution?

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

The required inner-product matrix – should be PSD.

	1	\mathbf{v}_{x_i}	\mathbf{v}_{y_i}	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$
1	1	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{x_i}	$1 - \delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$
\mathbf{v}_{y_i}	$1 - \delta$	$1 - 2\delta$	$1 - \delta$	$1 - 2\delta$	$1 - 2\delta$
$\mathbf{v}_{x_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$
$\mathbf{v}_{y_{i+1}}$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$	$1 - 2\delta$

The matrix is PSD because...

$$\begin{bmatrix} 1 & 1-\delta & 1-\delta & 1-2\delta & 1-2\delta \\ 1-\delta & 1-\delta & 1-2\delta & 1-2\delta & 1-2\delta \\ 1-\delta & 1-2\delta & 1-\delta & 1-2\delta & 1-2\delta \\ 1-2\delta & 1-2\delta & 1-2\delta & 1-2\delta & 1-2\delta \\ 1-2\delta & 1-2\delta & 1-2\delta & 1-2\delta & 1-2\delta \end{bmatrix}$$
$$= (1-2\delta) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \delta \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bad news. This is the only way to make the matrix PSD.

Why is that news bad?

From Step i

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

to Step $i + 1$

$x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$?
$0 \wedge 1 \rightarrow 0$?
$1 \wedge 0 \rightarrow 0$?

Why is that news bad?

From Step i

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$	δ
$1 \wedge 0 \rightarrow 0$	δ

to Step $i + 1$

$x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - 2\delta$
$0 \wedge 1 \rightarrow 0$?
$1 \wedge 0 \rightarrow 0$?

The inner-product $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}}$ is too large – angle between two vectors is 0.

The probability $\Pr[x_{i+2} = 1]$ ($\Pr[y_{i+2} = 1]$) cannot decrease.

Being less greedy – decrease the norm slower I

$x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$		$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow$	1	$1 - 1.2\delta$
$0 \wedge 1 \rightarrow$	0	0.2δ
$1 \wedge 0 \rightarrow$	0	0.2δ
$0 \wedge 0 \rightarrow$	1	0.1δ
$0 \wedge 0 \rightarrow$	0	0.7δ

The corresponding inner-product matrix.

	1	\mathbf{v}_{x_i}	\mathbf{v}_{y_i}	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$
1	1	$1 - \delta$	$1 - \delta$	$1 - 1.1\delta$	$1 - 1.1\delta$
\mathbf{v}_{x_i}	$1 - \delta$	$1 - \delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$
\mathbf{v}_{y_i}	$1 - \delta$	$1 - 1.2\delta$	$1 - \delta$	$1 - 1.2\delta$	$1 - 1.2\delta$
$\mathbf{v}_{x_{i+1}}$	$1 - 1.1\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.1\delta$	$1 - 1.2\delta$
$\mathbf{v}_{y_{i+1}}$	$1 - 1.1\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.1\delta$

The matrix is PSD because...

$$\begin{bmatrix} 1 & 1 - \delta & 1 - \delta & 1 - 1.1\delta & 1 - 1.1\delta \\ 1 - \delta & 1 - \delta & 1 - 1.2\delta & 1 - 1.2\delta & 1 - 1.2\delta \\ 1 - \delta & 1 - 1.2\delta & 1 - \delta & 1 - 1.2\delta & 1 - 1.2\delta \\ 1 - 1.1\delta & 1 - 1.2\delta & 1 - 1.2\delta & 1 - 1.1\delta & 1 - 1.2\delta \\ 1 - 1.1\delta & 1 - 1.2\delta & 1 - 1.2\delta & 1 - 1.2\delta & 1 - 1.1\delta \end{bmatrix} =$$
$$(1 - 1.2\delta) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \delta \begin{bmatrix} 1.2 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0 \\ 0.1 & 0 & & 0.1 & 0 \\ 0.1 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

Being less greedy – decrease the norm slower II

The corresponding inner-product matrix.

	1	\mathbf{v}_{x_i}	\mathbf{v}_{y_i}	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$
1	1	$1 - \delta$	$1 - \delta$	$1 - 1.1\delta$	$1 - 1.1\delta$
\mathbf{v}_{x_i}	$1 - \delta$	$1 - \delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$
\mathbf{v}_{y_i}	$1 - \delta$	$1 - 1.2\delta$	$1 - \delta$	$1 - 1.2\delta$	$1 - 1.2\delta$
$\mathbf{v}_{x_{i+1}}$	$1 - 1.1\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.1\delta$	$1 - 1.2\delta$
$\mathbf{v}_{y_{i+1}}$	$1 - 1.1\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.2\delta$	$1 - 1.1\delta$

Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - 1.1\delta = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\delta = 1 - 1.09\gamma$.

Would be good if $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\gamma$.

Amplify the angle I

Start point.

Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1 + \tau)\gamma$.

		$x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$		$\pi_C(\sigma)$	
		$1 \wedge 1 \rightarrow 1$	1	$1 - (1 + \tau)\gamma$	
		$0 \wedge 1 \rightarrow 0$	0	$\tau\gamma$	
		$1 \wedge 0 \rightarrow 0$	0	$\tau\gamma$	
		$0 \wedge 0 \rightarrow 1$	1	$\tau\gamma$	
		$0 \wedge 0 \rightarrow 0$	0	$(1 - 2\tau)\gamma$	

	I	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$	$\mathbf{v}_{x_{i+2}}$	$\mathbf{v}_{y_{i+2}}$
I	1	$1 - \gamma$	$1 - \gamma$	$1 - \gamma$	$1 - \gamma$
$\mathbf{v}_{x_{i+1}}$	$1 - \gamma$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$
$\mathbf{v}_{y_{i+1}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$
$\mathbf{v}_{x_{i+2}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - \gamma$	$1 - (1 + 1.5\tau)\gamma$
$\mathbf{v}_{y_{i+2}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + 1.5\tau)\gamma$	$1 - \gamma$

The matrix is PSD because...

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1-\gamma & 1-\gamma & 1-\gamma & 1-\gamma \\ 1-\gamma & 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma \\ 1-\gamma & 1-(1+\tau)\gamma & 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma \\ 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & 1-\gamma & 1-(1+1.5\tau)\gamma \\ 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & 1-(1+1.5\tau)\gamma & 1-\gamma \end{bmatrix} = \\
 & (1-(1+\tau)\gamma) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1+\tau & \tau & \tau & \tau & \tau \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ \tau & 0 & 0 & \tau & -0.5\tau \\ \tau & 0 & 0 & -0.5\tau & \tau \end{bmatrix}
 \end{aligned}$$

Where...

$$\begin{aligned} & \begin{bmatrix} 1+\tau & \tau & \tau & \tau & \tau \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ \tau & 0 & 0 & \tau & -0.5\tau \\ \tau & 0 & 0 & -0.5\tau & \tau \end{bmatrix} = \begin{bmatrix} 2\tau & \tau & \tau & 0 & 0 \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} 2\tau & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0.5\tau & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\tau & 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 & 0.5\tau \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5\tau & -0.5\tau \\ 0 & 0 & 0 & -0.5\tau & 0.5\tau \end{bmatrix} + \begin{bmatrix} 1-5\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

is PSD when $0 \leq \tau \leq 0.2$.

Amplify the angle II

Start point.

Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1 + \tau)\gamma$.

$x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$	$\pi_C(\sigma)$
$1 \wedge 1 \rightarrow 1$	$1 - (1 + \tau)\gamma$
$0 \wedge 1 \rightarrow 0$	$\tau\gamma$
$1 \wedge 0 \rightarrow 0$	$\tau\gamma$
$0 \wedge 0 \rightarrow 1$	$\tau\gamma$
$0 \wedge 0 \rightarrow 0$	$(1 - 2\tau)\gamma$

	I	$\mathbf{v}_{x_{i+1}}$	$\mathbf{v}_{y_{i+1}}$	$\mathbf{v}_{x_{i+2}}$	$\mathbf{v}_{y_{i+2}}$
I	1	$1 - \gamma$	$1 - \gamma$	$1 - \gamma$	$1 - \gamma$
$\mathbf{v}_{x_{i+1}}$	$1 - \gamma$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$
$\mathbf{v}_{y_{i+1}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$
$\mathbf{v}_{x_{i+2}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - \gamma$	$1 - (1 + 1.5\tau)\gamma$
$\mathbf{v}_{y_{i+2}}$	$1 - \gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + \tau)\gamma$	$1 - (1 + 1.5\tau)\gamma$	$1 - \gamma$

Result.

Norm: $\|\mathbf{v}_{x_{i+2}}\|^2 = \|\mathbf{v}_{y_{i+2}}\|^2 = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+2}} \cdot \mathbf{v}_{y_{i+2}} = 1 - (1 + 1.5\tau)\gamma$.

A two-stage block

Stage 1. Reduce the norm.

from	Norm: $\ \mathbf{v}_{x_i}\ ^2 = \ \mathbf{v}_{y_i}\ ^2 = 1 - \delta$ Inner-product: $\mathbf{v}_{x_i} \cdot \mathbf{v}_{y_i} = 1 - 1.2\delta$
------	--

Step i

to	Norm: $\ \mathbf{v}_{x_{i+1}}\ ^2 = \ \mathbf{v}_{y_{i+1}}\ ^2 = 1 - 1.1\delta = 1 - \gamma$ Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\delta = 1 - (1 + \tau)\gamma$
----	---

Stage 2. Amplify the angle (reduce the inner-product).

from	Norm: $\ \mathbf{v}_{x_{i+1}}\ ^2 = \ \mathbf{v}_{y_{i+1}}\ ^2 = 1 - \gamma$ Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1 + \tau)\gamma$
------	---

Step $i + 1$

to	Norm: $\ \mathbf{v}_{x_{i+2}}\ ^2 = \ \mathbf{v}_{y_{i+2}}\ ^2 = 1 - \gamma$ Inner-product: $\mathbf{v}_{x_{i+2}} \cdot \mathbf{v}_{y_{i+2}} = 1 - (1 + 1.5\tau)\gamma$
----	--

...

Step $i + (r - 1)$

to	Norm: $\ \mathbf{v}_{x_{i+r}}\ ^2 = \ \mathbf{v}_{y_{i+r}}\ ^2 = 1 - \gamma$ Inner-product: $\mathbf{v}_{x_{i+r}} \cdot \mathbf{v}_{y_{i+r}} = 1 - (1 + 1.5^{r-1}\tau)\gamma < 1 - 1.2\gamma$
----	--

Repeat the blocks

Suppose $k = qr$, let $\delta = 1.1^{-q}/1.2$.

Step 0	Norm: $\ \mathbf{v}_{x_0}\ ^2 = \ \mathbf{v}_{y_0}\ ^2 = 1 - \delta$ Inner-product: $\mathbf{v}_{x_0} \cdot \mathbf{v}_{y_0} = 1 - 1.2\delta$
Block 1	
Step r	Norm: $\ \mathbf{v}_{x_r}\ ^2 = \ \mathbf{v}_{y_r}\ ^2 = 1 - 1.1\delta$ Inner-product: $\mathbf{v}_{x_r} \cdot \mathbf{v}_{y_r} = 1 - 1.2 \cdot 1.1\delta$
Block 2	
Step $2r$	Norm: $\ \mathbf{v}_{x_{2r}}\ ^2 = \ \mathbf{v}_{y_{2r}}\ ^2 = 1 - 1.1^2\delta$ Inner-product: $\mathbf{v}_{x_{2r}} \cdot \mathbf{v}_{y_{2r}} = 1 - 1.2 \cdot 1.1^2\delta$
...	
Block q	
Step qr	Norm: $\ \mathbf{v}_{x_{qr}}\ ^2 = \ \mathbf{v}_{y_{qr}}\ ^2 = 1 - 1.1^q\delta$ Inner-product: $\mathbf{v}_{x_{qr}} \cdot \mathbf{v}_{y_{qr}} = 1 - 1.2 \cdot 1.1^q\delta = 0$
Step $k + 1 = qr + 1$	Norm: $\ \mathbf{v}_{x_{k+1}}\ ^2 = \ \mathbf{v}_{y_{k+1}}\ ^2 = 0$

Repeat the blocks

Suppose $k = qr$, let $\delta = 1.1^{-q}/1.2$.

Step 0	Norm: $\ \mathbf{v}_{x_0}\ ^2 = \ \mathbf{v}_{y_0}\ ^2 = 1 - \delta$ Inner-product: $\mathbf{v}_{x_0} \cdot \mathbf{v}_{y_0} = 1 - 1.2\delta$
Block 1	
Step r	Norm: $\ \mathbf{v}_{x_r}\ ^2 = \ \mathbf{v}_{y_r}\ ^2 = 1 - 1.1\delta$ Inner-product: $\mathbf{v}_{x_r} \cdot \mathbf{v}_{y_r} = 1 - 1.2 \cdot 1.1\delta$
Block 2	
Step $2r$	Norm: $\ \mathbf{v}_{x_{2r}}\ ^2 = \ \mathbf{v}_{y_{2r}}\ ^2 = 1 - 1.1^2\delta$ Inner-product: $\mathbf{v}_{x_{2r}} \cdot \mathbf{v}_{y_{2r}} = 1 - 1.2 \cdot 1.1^2\delta$
...	
Block q	
Step qr	Norm: $\ \mathbf{v}_{x_{qr}}\ ^2 = \ \mathbf{v}_{y_{qr}}\ ^2 = 1 - 1.1^q\delta$ Inner-product: $\mathbf{v}_{x_{qr}} \cdot \mathbf{v}_{y_{qr}} = 1 - 1.2 \cdot 1.1^q\delta = 0$
Step $k + 1 = qr + 1$	Norm: $\ \mathbf{v}_{x_{k+1}}\ ^2 = \ \mathbf{v}_{y_{k+1}}\ ^2 = 0$

loss = $2\delta = 2^{-\Omega(k)}$ only from Step 0.

The SDP gap

Gap instance $\mathcal{I}_k^{\text{Horn}}$.

$$\begin{aligned} \text{Step 0:} & & x_0, & y_0 \\ \text{Step 1:} & & x_0 \wedge y_0 \rightarrow x_1, & x_0 \wedge y_0 \rightarrow y_1 \\ \text{Step 2:} & & x_1 \wedge y_1 \rightarrow x_2, & x_1 \wedge y_1 \rightarrow y_2 \\ \text{Step 3:} & & x_2 \wedge y_2 \rightarrow x_3, & x_2 \wedge y_2 \rightarrow y_3 \\ & & \dots & \\ \text{Step } k+1: & & x_k \wedge y_k \rightarrow x_{k+1}, & x_k \wedge y_k \rightarrow y_{k+1} \\ \text{Step } k+2: & & \overline{x_{k+1}}, & \overline{y_{k+1}} \end{aligned}$$

Observation

$\mathcal{I}_k^{\text{Horn}}$ is not satisfiable. Therefore $\text{OPT}(\mathcal{I}_k^{\text{Horn}}) \leq 1 - \Omega(1/k)$.

Lemma

$\text{OPT}_{\text{SDP}}(\mathcal{I}_k^{\text{Horn}}) \geq 1 - 2^{-\Omega(k)}$.

Thank you

Questions?



Venkatesan Guruswami, and Yuan Zhou, *Tight Bounds on the Approximability of Almost-satisfiable Horn SAT and Exact Hitting Set*, SODA 2011 (to appear).