# Optimal strong parallel repetition for projection games on low threshold rank graphs

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**Abstract.** Given a two-player one-round game $G$ with value $\mathrm{val}(G) = (1-\eta)$, how quickly does the value decay under parallel repetition? If $G$ is a projection game, then it is known that we can guarantee $\mathrm{val}(G^{\otimes n}) \le (1-\eta^2)^{\Omega(n)}$, and that this is optimal. An important question is under what conditions can we guarantee that *strong* parallel repetition holds, i.e. $\mathrm{val}(G^{\otimes}) \le (1-\eta)^{\Omega(n)}$?
In this work, we show a strong parallel repetition theorem for the case when $G$'s constraint graph has low threshold rank. In particular, for any $k \ge 2$, if $\sigma_k$ is the $k$-th largest singular value of $G$'s constraint graph, then we show that

$$\mathrm{val}(G^{\otimes n}) \le \left(1 - \frac{\sqrt{1-\sigma_k^2}}{k} \cdot \eta\right)^{\Omega(n)}.$$

When $k$ is a constant and $\sigma_k$ is bounded away from 1, this decays like $(1-\eta)^{\Omega(n)}$. In addition, we show that this upper-bound exactly matches the value of the Odd-Cycle Game under parallel repetition. As a result, our parallel repetition theorem is tight.
This improves and generalizes upon the work of [RR12], who showed a strong parallel repetition theorem for the case when $G$'s constraint graph is an expander.

## 1 Introduction

A *two-prover one-round game* $G$ on questions $U \cup V$ is a distribution on $(u, v)$ (where $u \in U, v \in V$) so that $u$ is given to the first prover and $v$ is given to the second prover. The provers respond with answers $f(u)$ and $g(v)$, which are from the set $\Sigma$. The answers are accepted if they satisfy the predicate $\pi_{u,v}(f(u), g(v))$ associated to the pair of questions $(u, v)$. We say $G$ is a *projection game* if for every predicate $\pi_{u,v}$ and every $\beta \in \Sigma$, there is at most one $\alpha \in \Sigma$ such that $\pi_{u,v}(\alpha, \beta)$ is satisfied. The *constraint graph* of $G$ is the bipartite graph $H$ with vertex set $U \cup V$ corresponding to the distribution of questions. For the precise definitions, please refer to Section 2.

Perhaps the most fundamental result in the area of two-prover one-round games is the parallel repetition theorem of Raz [Raz98]. In its version for projection games by Rao [Rao11], it states:

**Theorem 1.** *Let $G$ be a projection game. If $\mathrm{val}(G) \le 1-\eta$, then $\mathrm{val}(G^{\otimes n}) \le (1-\eta^2)^{\Omega(n)}$.*

Stated contrapositively, if $\mathrm{val}(G^{\otimes n}) \ge (1-\eta)^n$, then $\mathrm{val}(G) \ge 1 - O(\sqrt{\eta})$. Naively, one would expect a faster rate of decay: that is, if $\mathrm{val}(G) \le 1-\eta$, then $\mathrm{val}(G^{\otimes n})$ should satisfy $\mathrm{val}(G^{\otimes n}) \le (1-\eta)^{\Omega(n)}$. A parallel repetition bound of this form is known as *strong* parallel repetition, and it was open whether a strong parallel repetition theorem was true in general, or even for interesting restricted classes of games (e.g. Unique Games; see [FKO07]), until Raz [Raz11] showed that strong parallel repetition fails on the so-called Odd Cycle Game. In particular, he showed that $\mathrm{val}(G^{\otimes n})$ for the Odd Cycle Game matches exactly the upper bound given in Theorem 1. Thus, Theorem 1 is essentially the best parallel repetition theorem one can show for general projection games. As the Odd Cycle Game has almost every nice property one could hope for in a two-player one-round game, this closed the door on a strong parallel repetition theorem for many interesting subclasses of two-player one-round games as well.

One interesting subclass not covered by the Odd Cycle Game is the class of *expanding* games. This is the class of two-player one-round games $G$ in which the constraint graph of $G$ is an expander. This is an interesting class of two-player one-round games which appears frequently in the hardness of approximation literature. Following the work of [AKK+08] and [BRR+09], Raz and Rosen [RR12] showed a strong parallel repetition theorem for expanding games. The following exact bound is implicit in their paper:

**Theorem 2.** *Let $G$ be a projection game, and let $\sigma_2$ be the second largest singular value of $G$'s constraint graph. If* $\mathrm{val}(G) \leq (1-\eta)$*, then* $\mathrm{val}(G^{\otimes n}) \leq (1-(1-\sigma_2)^2 \cdot \eta)^{\Omega(n)}$.

In particular, if $\sigma_2$ is a constant, then $\mathrm{val}(G^{\otimes n})$ decays like $(1-\eta)^{\Omega(n)}$.

This result motivates a couple of questions. The first is obvious: is the dependence on $\sigma_2$ in Theorem 2 tight? If not, can it be improved? For the second question, we begin with a definition. A graph with *low threshold rank* is one whose $k$-th largest singular value $\sigma_k$ is bounded away from 1, where $k$ is small (say, a constant). The study of low threshold rank graphs was initiated in [Kol11], and they were first formally defined in [ABS10]. Since then, they, and the related class of *small set expanders*, have taken on an increasingly important role in the field of approximation algorithms (see, e.g., [GS11,RST12,GT13]). Thus, our second question is the following: does a strong parallel repetition theorem hold for graphs with low threshold rank?

### 1.1 Parallel repetition and Cheeger's inequality

In this work, we answer both of these questions using the new parallel repetition framework of [DS13]. They study a relaxation to the value of the game, called $\mathrm{val}_+(G)$, which, roughly speaking, is the best value achieved on the game by a *distribution* of "fractional assignments". These "fractional assignments" are required to collectively look like a valid assignment, but individually they may look very different than a valid assignment. This relaxation $\mathrm{val}_+(G)$ enjoys several nice analytic properties: for example, it strictly upper-bounds $\mathrm{val}(G)$, and it is multiplicative under parallel repetition. From here, they give a new proof of Theorem 1 using the following three-step process:

1. Supposing $\mathrm{val}(G^{\otimes n}) \geq (1-\eta)^n$, then $\mathrm{val}_+(G) \geq (1-\eta)$. This gives a distribution of "fractional assignments" with "fractional value" at least $(1-\eta)$.
2. Round each "fractional assignment" to a 0/1-assignment using Cheeger rounding.
3. Combine these 0/1-assignments into a single 0/1 assignment using the correlated sampling approach of [BHH+08].

Supposing that $\mathrm{val}(G^{\otimes n}) \geq (1-\eta)^n$, then this will produce a solution to $G$ with value $(1-O(\sqrt{\eta}))$.

This proof has revealed a strong connection between the parallel repetition theorem and Cheeger's inequality. To explain this, we begin with some necessary definitions. Given a graph $H = (V, E)$, the *conductance* of a set $S \subseteq V$ is

$$\phi(S) = \frac{|E(S, \overline{S})|}{d \cdot \min\{|S|, |\overline{S}|\}},$$

where $E(S, \overline{S})$ is the set of edges in $H$ which cross from $S$ to $\overline{S}$. Given this, we can define the conductance of $H$ to be the worst-case conductance over all subsets $S$, i.e. $\phi(H) := \min_{S \subset V}(\phi(S))$. Cheeger's inequality states:

**Cheeger's inequality.** *Given a graph $H = (V, E)$, let $\lambda_2$ be the second-smallest eigenvalue of its normalized Laplacian. Then*

$$\frac{1}{2}\lambda_2 \leq \phi(G) \leq \sqrt{2\lambda_2}.$$

The upper-bound $\phi(G) \leq \sqrt{2\lambda_2}$ is shown by taking the second-largest eigenvector $v_2$, which has a "fractional conductance" of $\lambda_2$, and performing a process called "Cheeger rounding" on it to produce a set $S$ of conductance at most $\sqrt{2\lambda_2}$. This is the same Cheeger rounding that takes place in step 2 in the parallel repetition proof, and is the reason that we can only prove $\text{val}(G) \geq (1-\sqrt{\eta})$ if $\text{val}_+(G) \geq (1-\eta)$. In both Cheeger's inequality and in the parallel repetition theorem, we "lose a square root", and in both cases this happens for the same reason.

Thus, to prove a strong parallel repetition theorem, it seems we need a "strong" Cheeger's inequality, one that rounds vectors of "fractional conductance" $\eta$ to sets of conductance $O(\eta)$. Recently, a whole host of works have been published which show how to modify/improve Cheeger's inequality to account for the higher eigenvalues of the graph (for just a tip of the iceberg, see [LOGT12] and [LRTV12]). One of these is indeed a "strong" Cheeger's inequality for the case when the graph has low threshold rank [KLL+13]:

**Theorem 3.** *Given a graph $H = (V, E)$, let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|}$ be the eigenvalues of its normalized Laplacian. Then for every $k \geq 2$,*

$$\phi(H) \leq O(k)\frac{\lambda_2}{\sqrt{\lambda_k}}.$$

In particular, if $\lambda_k$ is large for $k$ a constant (in other words, if $H$ has low threshold rank), then an eigenvector of $H$ with eigenvalue $\lambda_2$ can be rounded into a cut of sparsity $\approx \lambda_2$. This theorem gives hope for a strong parallel repetition theorem for the low threshold rank case.

### 1.2 Our results

We may now state our main theorem.

**Theorem 4.** *Let $G$ be a projection game, and let $1 = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{|V|}$ be the singular values of $G$'s constraint graph. For any $k \geq 2$, if $\text{val}(G) \leq (1-\eta)$, then*

$$\text{val}(G^{\otimes n}) \leq \left(1 - \frac{\sqrt{1-\sigma_k^2}}{k} \cdot \eta\right)^{\Omega(n)}.$$

At a high level, our proof of this comes from combining Theorem 3 with the parallel repetition framework of [DS13]. However, the interface between these two isn't clean, and some care must be taken in combining them. Furthermore, we can't just apply Theorem 3 all at once; instead, we have to wait to apply its various components at the appropriate time (at a high level, this is because whereas the normal Cheeger rounding preservers marginals, this higher-order Cheeger rounding only *approximately* preserves marginals).

In addition, we give another proof of strong parallel repetition for the special case of expanding games. In other words, we reprove Theorem 4 in the case of $k = 2$, except the bound we get is

$$\text{val}(G^{\otimes n}) \leq \left(1 - (1-\sigma_2^2) \cdot \eta\right)^{\Omega(n)}.$$

While this is (slightly) quantitatively worse, the proof is much easier: everything is elementary, and the entire proof essentially rests on one well-timed application of Cauchy-Schwarz.

### 1.3 Tightness of our results

Our Theorem 4 is tight, as certified by the Odd Cycle Game. This should not be too surprising: the Odd Cycle Game is tight for Theorem 1, and the cycle graph is tight for both the normal Cheeger's inequality and the "strong" Cheeger's inequality of Theorem 3. To see that the Odd Cycle Game is a tight example for our theorem, we begin with a description of it.

Let $k \geq 0$, and set $m := 2k+1$. Let $G_m = (V_m, E_m)$ be the cycle on $m$ vertices. In the Odd Cycle Game, the two players $\mathsf{P1}$ and $\mathsf{P2}$ are trying to convince the verifier that the graph $G_m$ is 2-colorable. Formally, each player $\mathsf{Pi}$ is given as questions the vertices in $V_m$, and verifier expects their labels to be in the set $\{0,1\}$. The constraints are distributed as follows:

- With probability $1/2$, a vertex $v \in V_m$ is chosen uniformly at random and given to both players. Player $\mathsf{Pi}$ responds with $b_i \in \{0,1\}$, and the constraint is that $b_1 = b_2$.
- With probability $1/2$, an edge $(v_1, v_2) \in E_m$ is selected uniformly at random; $\mathsf{P1}$ is given $v_1$, and $\mathsf{P2}$ is given $v_2$. Player $\mathsf{Pi}$ responds with $b_i \in \{0,1\}$, and the constraint is that $b_1 \neq b_2$.

It is easy to see that $\mathrm{val}(G) = 1 - \frac{1}{2m}$. In [Raz11], Raz determined how the value of $G$ changes under parallel repetition:

**Theorem 5.** *For large enough $n$,* $\mathrm{val}(G^{\otimes n}) \geq \left(1 - \frac{1}{m^2}\right)^{O(n)}$.

Let us now calculate the bound our Theorem 4 gives for the Odd Cycle Game. The constraint graph of the Odd Cycle Game is w/prob. 1/2 the identity mapping and w/prob. 1/2 a random step on the cycle graph. As the cycle graph has eigenvalues $\cos(2\pi k/m)$ for each $k \in \{0, \ldots, m-1\}$ (see, for example, [Tre11]) the constraint graph of the Odd Cycle Game has singular values $1/2 + \cos(2\pi k/m)/2$ for each $k \in \{0, \ldots, m-1\}$. For small values of $k$, this means that the $k$-th largest singular value of the constraint graph is $\approx 1 - \left(\frac{k}{m}\right)^2$. Plugging this into our Theorem 4, we see that it gives a guarantee of

$$\mathrm{val}(G^{\otimes n}) \leq \left(1 - \frac{1}{m^2}\right)^{\Omega(n)},$$

exactly matching Theorem 5.

### 1.4 Organization

Definitions and preliminary materials can be found in Section 2. Our proof of Theorem 4 can be found in Section 3. The simple proof of strong parallel repetition for expanding games can be found in Section 4.

## 2 Preliminaries

*Two-prover one-round games.* A two-prover one-round game $G$ is associated with a bipartite graph $H = (U, V, E)$ (known as the *constraint graph*) and a set $\Sigma$ of answers. Each edge $(u,v) \in E$ is associated with a predicate $\pi_{u,v} : \Sigma \times \Sigma \to \{\mathsf{True}, \mathsf{False}\}$. The prover choses a random edge $(u,v)$ uniformly from $E$ and send $u$, $v$ to the two provers respectively. The provers respond with answers $f(u)$ and $g(v)$, from the set $\Sigma$. The answers are accepted if $\pi_{u,v}(f(u), g(v))$ is satisfied. We say $G$ is a *projection game* if for every predicate $\pi_{u,v}$ and every $\beta \in \Sigma$, there is at most one $\alpha \in \Sigma$ such that $\pi_{u,v}(\alpha, \beta)$ is satisfied. By $\mathrm{val}(G)$ we denote the best value achievable by functions $f$ and $g$.

Given two two-prover one-round games $G$ and $H$, we denote the parallel repetition of $G_1$ and $G_2$ by $G_1 \otimes G_2$. This is the two-prover one-round game which is played as follows:

1. Sample $(u_1, v_1) \sim G_1$ and $(u_2, v_2) \sim G_2$.
2. Give question $(u_1, u_2)$ to P1 and $(v_1, v_2)$ to P2.
3. Receive answers $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$.
4. Accept iff $\pi_{u_1,v_1}(\alpha_1, \beta_1) = \pi_{u_2,v_2}(\alpha_2, \beta_2) = 1$.

*Matrices and vectors.* Vectors will be indexed in one of three ways: either by vertices $v \in V$, by labels $\beta \in \Sigma$, or by vertex/label pairs $(v, \beta) \in V \times \Sigma$. As in [DS13], inner products will use the uniform probability measure on vertices and the counting measure on labels. Thus, if $f$ and $g$ are of the form $f, g : V \to \mathbb{R}$, then

$$\langle f, g\rangle = \mathop{\mathbf{E}}_{v \in V} f(v) \cdot g(v);$$

if $f$ and $g$ are of the form $f, g : \Sigma \to \mathbb{R}$, then

$$\langle f, g\rangle = \sum_{\beta \in \Sigma} f(\beta) \cdot g(\beta);$$

and if $f$ and $g$ are of the form $f, g : V \times \Sigma \to \mathbb{R}$, then

$$\langle f, g\rangle = \mathop{\mathbf{E}}_{v \in V} \sum_{\beta \in \Sigma} f(v, \beta) \cdot g(v, \beta).$$

Given $f : V \times \Sigma \to \mathbb{R}$ and a vertex $v \in V$, we will write $f(v, \cdot)$ for the function mapping $\Sigma \to \mathbb{R}$ which, on input $\beta \in \Sigma$, outputs $f(v, \beta)$. Essentially every function with domain $\Sigma$ in this paper will be produced in this fashion.

Now we define the matrix/vector and matrix/matrix products in accordance with these inner products. If $A$ is a $|U \times \Sigma|$-by-$|V \times \Sigma|$ matrix and $g : V \times \Sigma \to \mathbb{R}$ is a vector, then the matrix/vector product $Ag : U \times \Sigma \to \mathbb{R}$ is the vector for which $(Ag)(u, \alpha) = \langle A_{(u,\alpha)}, g\rangle$, where $A_{(u,\alpha)}$ is the $(u, \alpha)$-th row of $A$. Similarly, if $f : U \times \Sigma \to \mathbb{R}$ is a vector, then the matrix/vector product $f^\top A : V \times \Sigma \to \mathbb{R}$ is the (transposed) vector for which $(f^\top A)(v, \beta) = \langle f, a_{(v,\beta)}\rangle$, where $a_{(v,\beta)}$ is the $(v, \beta)$-th column of $A$. By setting up the matrix/vector product in this way, we get the following set of equalities:

$$\langle f, Ag\rangle = f^\top Ag = \langle A^\top f, g\rangle.$$

If $A$ and $B$ are $|U \times \Sigma|$-by-$|V \times \Sigma|$ matrices, then the matrix $A^\top B$ can be defined as follows:

$$(AB)_{(v_1,\beta_1),(v_2,\beta_2)} = (a_{(v_1,\beta_1)}, b_{(v_2,\beta_2)}),$$

where $a_{(v_1,\sigma_1)}$ is the $(v_1, \sigma_1)$-th column of $A$ and $b_{(v_2,\sigma_2)}$ is the $(v_2, \sigma_2)$-th column of $B$. Finally, given two vectors $f : (U \times \Sigma) \to \mathbb{R}$ and $g : (V \times \Sigma) \to \mathbb{R}$, the outer product $f \cdot g^\top$ is simply the matrix in which $(f, g^\top)_{(u,\alpha),(v,\beta)} = f_{(u,\alpha)} \cdot g_{(v,\beta)}$.

We note that these matrix products were defined only for matrices whose indices are of the form $V \times \Sigma$. However, as the above definitions only used the inner product defined on $V \times \Sigma$, similar products can be defined for matrices with indices of the form $V$ or $\Sigma$ using the inner products on those spaces.

*The projection game operator.* We will associate with $G$ a linear operator (also named $G$) which projects assignments $g : V \times \Sigma \to \mathbb{R}$ onto assignments of type $U \times \Sigma \to \mathbb{R}$. To begin, fix adjacent vertices $u$ and $v$. As previously defined, $g(v, \cdot)$ is the function from $\Sigma$ to $\mathbb{R}$ denoting the part of $g$ restricted to $v$. Write $G_{u\leftarrow v}$ for the operator which projects assignments for $v$ onto assignments for $u$. In other words,

$$\left(G_{u\leftarrow v}\, g(v, \cdot)\right)(\alpha) = \sum_{\beta : \pi_{uv}(\beta) = \alpha} g(v, \beta).$$

Now we can define the action of the projection game operator $G$ on $g$ as follows:

$$(Gg)(u,\alpha) = \mathop{\mathbf{E}}_{v\sim u} \left(G_{u\leftarrow v} g(v,\cdot)\right)(\alpha) \tag{1}$$
$$= \mathop{\mathbf{E}}_{v\sim u} \sum_{\beta:\pi_{uv}(\alpha,\beta)=1} g(v,\beta).$$

To see why this is a natural operator to represent the game, let $f : U \to \Sigma$ and $g : V \to \Sigma$ be assignments. Overloading notation, write $f : U \times \Sigma \to \mathbb{R}$ and $g : V \times \Sigma \to \mathbb{R}$ for the 0/1-valued vectors which correspond to $f$ and $g$. Then we have that

$$\mathrm{val}(G; f, g) = f^\top G g = \langle f, Gg\rangle.$$

Finally, though we will not need this, we note that the operators $G_{u\leftarrow v}$ and $G$ can be realized as matrices in the following way:

$$(G_{u\leftarrow v})_{\alpha,\beta} = \begin{cases} 1 \text{ if } \pi_{u,v}(\alpha,\beta) = 1, \\ 0 \text{ otherwise,} \end{cases}$$

and

$$G_{(u,\alpha),(v,\beta)} = \begin{cases} \frac{|V|}{d_U} \text{ if } u \sim v \text{ and } \pi_{u,v}(\alpha,\beta) = 1, \\ 0 \text{ otherwise,} \end{cases}$$

where $d_U$ is the degree of the vertices in $U$.

*Quadratic forms.* We will be interested the quadratic form $\|Gg\|^2$, where $g$ is of the form $g : V \times \Sigma \to \mathbb{R}$. The exact expression for this quadratic form is somewhat cumbersome to work with, but by appealing to Equation 1, we have the following expression, which will we use repeatedly.

$$\|Gg\|^2 = \mathop{\mathbf{E}}_{u\in U} \mathop{\mathbf{E}}_{v_1,v_2\sim u} \langle G_{u\leftarrow v_1} g(v_1,\cdot), G_{u\leftarrow v_2} g(v_2,\cdot)\rangle. \tag{2}$$

Here $v_1, v_2 \sim u$ means that $v_1$ and $v_2$ are independent, uniformly random neighbors of $u$. This also shows that $G$ is norm-reducing on fractional assignments, as for any fractional assignment $g$ and vertices $u$, $v_1$, and $v_2$,

$$\langle G_{u\leftarrow v_1} g(v_1,\cdot), G_{u\leftarrow v_2} g(v_2,\cdot)\rangle \leq \|G_{u\leftarrow v_1} g(v_1,\cdot)\| \cdot \|G_{u\leftarrow v_2} g(v_2,\cdot)\|$$
$$\leq \frac{\|G_{u\leftarrow v_1} g(v_1,\cdot)\|^2 + \|G_{u\leftarrow v_2} g(v_2,\cdot)\|^2}{2}.$$

Thus, by Equation (2)

$$\|Gg\|^2 \leq \mathop{\mathbf{E}}_{u\in U} \mathop{\mathbf{E}}_{v_1,v_2\sim u} \frac{\|G_{u\leftarrow v_1} g(v_1,\cdot)\|^2 + \|G_{u\leftarrow v_2} g(v_2,\cdot)\|^2}{2} = \|g\|^2,$$

where the last line follows because $H$ is biregular. Note that this is not necessarily true when $g$ is not a fractional assignment (in particular, the equality need not hold).

*Projection games relaxation.* The paper of [DS13] introduced a convenient relaxation for the value of the game $G$. Before stating it, let us fix some notation. Because the game is bipartite, it is convenient to use two different functions $f : U \times \Sigma \to \mathbb{R}$ and $g : V \times \Sigma \to \mathbb{R}$ to label the two vertex sets. We will reserve $\alpha \in \Sigma$ to refer to labels on the $U$ side and $\beta \in \Sigma$ to refer to labels on the $V$ side. An important subset of functions for us will be the *fractional assignments*. We say that $f$ is a fractional assignment if for each $u \in U$, $f(u, \alpha)$ is nonzero for at most one $\alpha$ (a similar definition holds for $g$).

A (usually) finite measure space is a finite set $\Omega$ along with a measure $\mu : \Omega \to \mathbb{R}^{\geq 0}$ on that set. Typically, we will keep the set explicit while the measure will be implicit. Given a quantity $q : \Omega \to \mathbb{R}$, we define the expectation with respect to $\Omega$ as

$$\mathop{\mathbf{E}}_{\omega \sim \Omega}[q(\omega)] = \sum_{\omega \in \Omega} \mu(\omega) \cdot q(\omega).$$

Now, we can specify the relaxation. The measure space $\Omega$ is allowed to be arbitrary.

**Projection games relaxation:**

$$\begin{array}{rl} \textbf{maximize} & \mathrm{val}_+(f,g) := \mathop{\mathbf{E}}_{\omega \sim \Omega} \langle f_\omega, G g_\omega \rangle \\ \textbf{subject to} & f_\omega \text{ and } g_\omega \text{ are fractional assignments, for all } \omega \in \Omega \\ & \mathop{\mathbf{E}}_{\omega \sim \Omega} \|f_\omega(u, \cdot)\|^2 \leq 1, \quad \forall u \\ & \mathop{\mathbf{E}}_{\omega \sim \Omega} \|g_\omega(v, \cdot)\|^2 \leq 1, \quad \forall v \\ & f, g \geq 0 \end{array}$$

Let $\mathrm{val}_+(G)$ denote the value of the relaxation. We have the following two results, both from [DS13]:

**Proposition 1.** *Let $G$ be a game. Then* $\mathrm{val}_+(G) \geq \mathrm{val}(G)$.

**Theorem 6.** *Let $G_1$ and $G_2$ two bipartite projection games and let $G_1' = G_1^\top G_1$ and $G_2' = G_2^\top G_2$. Then*

$$\mathrm{val}_+(G_1' \otimes G_2') = \mathrm{val}_+(G_1') \cdot \mathrm{val}_+(G_2').$$

Combining these two with Claim 2.1 of [DS13], we see that

**Proposition 2.** *Let $G$ be a bipartite projection game. Then* $\mathrm{val}(G^{\otimes k}) \leq \mathrm{val}_+(G)^{k/2}$.

Sometimes the full power of the relaxation isn't needed. In these cases, we can use the following lemma:

**Lemma 1.** *If there exists an assignment $f$ such that $\|G^{\otimes k} f\|^2 \geq (1-\eta)^k$, then there exists a fractional assignment $g : V \times \Sigma \to \mathbb{R}$ such that $\|Gg\|^2 \geq 1 - \eta$.*

*The constraint graph.* A primary focus for us in this paper will be the constraint graph $H$ and its spectrum. Like $G$, we can represent $H$ as a linear operator which projects vectors $h : V \to \mathbb{R}$ onto assignments of type $U \to \mathbb{R}$, as follows:

$$(Hh)(u) = \mathop{\mathbf{E}}_{v \sim u} h(v).$$

We can represent $H$ as a $|U| \times |V|$ matrix (which we will also refer to as $H$), defined as follows:

$$H_{u,v} = \begin{cases} \frac{|V|}{d_U} \text{ if } u \sim v, \\ 0 \text{ otherwise,} \end{cases}$$

where $d_U$ is the degree of the vertices in $U$.

For the spectrum of $H$ we will use the SVD. Suppose the rank of $H$ is $d$, and let $l_1, \ldots, l_d$ be its left-singular vectors, $r_1, \ldots, r_d$ be its right-singular vectors, and $\sigma_1 \geq \ldots \geq \sigma_d$ be its singular values. Then we can write

$$H = \sum_{i=1}^{d} \sigma_i \cdot l_i \cdot r_i^\top. \tag{3}$$

Throughout, we will assume that $H$ is biregular. As a result, $\sigma_1 = 1$, $l_1$ and $r_1$ are the constantly-one vectors, and $\sigma_i \in [0, 1]$ for all $i$. In addition, the graph on $V$ corresponding to $H^\top H$ is also regular.

*Constraint graph assignments.* If $g : V \times \Sigma \to \mathbb{R}^{\geq 0}$ is a fractional assignment, then there is at most one $\beta \in \Sigma$ such that $g(v, \beta)$ is nonzero, for each $v \in V$. Thus, we can write $g(v, \beta) = h(v) \cdot I(v, \beta)$, where $h : V \to \mathbb{R}$ and $I(\cdot, \cdot)$ is a 0/1 indicator. For convenience, we will require that for each $v \in V$, there exists an $i \in \Sigma$ such that $I(v, \beta) = 1$. As a result, we can write $h$ as $h(v) = \sum_{\beta \in \Sigma} g(v, \beta)$.

Some properties of $g$ carry over to $h$. First, because $g$ is a fractional assignment,

$$\|h\|^2 := \mathop{\mathbf{E}}_{v \in V} \left[h(v)^2\right] = \mathop{\mathbf{E}}_{v \in V} \left[\sum_{\beta \in \Sigma} g(v, \beta)^2\right] = \|g\|^2 = 1.$$

Next, using $H$ as the constraint graph of $G$, we have that $(Hh)(u) = \sum_{\alpha \in \Sigma} (Gg)(u, \alpha)$. As a result,

$$(Hh)(u)^2 = \left(\sum_{\alpha \in \Sigma} (Gg)(u, \alpha)\right)^2 \geq \sum_{\alpha \in \Sigma} (Gg)(u, \alpha)^2,$$

where the inequality follows from the nonnegativity of $g$. Taking an expectation over $u$, this means that $\|Hh\|^2 \geq \|Gg\|^2$.

*Operators.* So as not to conflict with the above notation, we will write $Id$ for the identity operator. We note that because we are using nonstandard matrix multiplication, the matrix representing the $Id$ operator may have some number other than one on its diagonal.

## 3 Parallel repetition for low threshold rank graphs

In this section, we prove the following theorem.

**Theorem 7.** *Let $G$ be a projection game over vertex set $U \cup V$. Given $k \geq 2$, let $\sigma_k$ be the $k$-th singular value of $G$'s constraint graph $H$. If $\mathcal{S} = (f, g, \Omega)$ is a solution to the relaxation with* $\mathrm{val}_+(\mathcal{S}) \geq 1 - \eta$, *then*

$$\mathrm{val}(G) \geq 1 - \frac{64 k \eta}{\sqrt{1 - \sigma_k^2}}.$$

Combining this with Proposition 2 yields Theorem 4.

*Proof.* As $\mathrm{val}_+(\mathcal{S})$ is monotically increasing in $f$ and $g$, we can assume that

$$\mathop{\mathbf{E}}_{\omega \sim \Omega} \|f_\omega\|^2 = \mathop{\mathbf{E}}_{\omega \sim \Omega} \|g_\omega\|^2 = 1.$$

In particular, this means that for each $\alpha, \beta \in \Sigma$,

$$\mathop{\mathbf{E}}_{\omega \sim \Omega} \|f_\omega(\alpha, \cdot)\|^2 = \mathop{\mathbf{E}}_{\omega \sim \Omega} \|g_\omega(\beta, \cdot)\|^2 = 1.$$

We begin our proof by focusing in on $g$.

**Proposition 3.** $\mathbf{E}_{\omega\sim\Omega} \|Gg_\omega\|^2 \geq 1 - 2\eta$.

*Proof.* By Cauchy-Schwarz,

$$1-\eta \leq \mathop{\mathbf{E}}_{\omega\sim\Omega} \langle f_\omega, Gg_\omega\rangle \leq \mathop{\mathbf{E}}_{\omega} \|f_\omega\| \cdot \|Gg_\omega\| \leq \frac{\mathbf{E}_\omega \|f_\omega\|^2 + \mathbf{E}_\omega \|Gg_\omega\|^2}{2}.$$

Since $\mathbf{E}\,\|f_\omega\|^2 = 1$, this inequality is satisfied only if $\mathbf{E}\,\|Gg_\omega\|^2 \geq 1-2\eta$.

Next, we show how to convert $g$ into a 0/1-valued solution, albeit one with a lower value.

**Lemma 2.** *There is a 0/1 valued solution $(g'_\omega, \Omega')$ with*

$$\mathop{\mathbf{E}}_{\omega\sim\Omega'} \|Gg'_\omega\|^2 \geq \mathop{\mathbf{E}}_{\omega\sim\Omega'} \|g'_\omega\|^2 - \frac{4\eta}{\sqrt{1-\sigma_k^2}}.$$

*Furthermore, $\frac{1}{8k} \leq \mathbf{E}_{\omega\sim\Omega'} \|g'_\omega(v,\cdot)\|^2$, for each $v \in V$.*

*Proof.* For a fixed $\omega \in \Omega$, $g_\omega$ is a fractional assignment for the $V$ vertices in $G$. We will "round" each $g_\omega$ independently using the following lemma.

**Lemma 3.** *Let $g : V \times \Sigma \to \mathbb{R}^{\geq 0}$ be a fractional assignment for $G$. Then there exists a measure $\mu : [0,\infty) \to \mathbb{R}^{\geq 0}$ and a 0/1-assignment $g^{(t)} : V \times \Sigma \to \{0,1\}$, for each $t \geq 0$, such that*

$$\mathop{\mathbf{E}}_{t\sim\mu} \langle g^{(t)}, (Id - G^\top G) g^{(t)}\rangle \leq \left(\frac{1}{2} + \frac{\sqrt{2}}{\sqrt{1-\sigma_k^2}}\right) \langle g, (Id - G^\top G) g\rangle,$$

*Furthermore, $\frac{1}{8k} g(v,\beta)^2 \leq \mathbf{E}_{t\sim\mu}\, g^{(t)}(v,\beta)^2 \leq g(v,\beta)^2$ , for each $v \in V, \beta \in \Sigma$.*

This lemma is based on the improved Cheeger's inequality from [KLL+13]. We defer the proof to Appendix A.

For each $\omega \in \Omega$, apply Lemma 3 to $g_\omega$. Let $\mu_\omega$ be the resulting measure and $g_\omega^{(t)}$ be the resulting 0/1-assignments. Now, set $\Omega' := \Omega \times [0,\infty)$, and let its corresponding measure be distributed as follows: sample $\omega \sim \Omega$ and $t \sim \mu_\omega$, and output $(\omega, t)$. The 0/1-valued solution we will use is $(g_\omega^{(t)}, \Omega')$. Taking the expectation over $\omega$, Lemma 3 guarantees that

$$\frac{1}{8k} \mathop{\mathbf{E}}_{\omega\sim\Omega} g_\omega(v,\beta)^2 \leq \mathop{\mathbf{E}}_{\omega\sim\Omega} \mathop{\mathbf{E}}_{t\sim\mu_\omega} g_\omega^{(t)}(v,\beta)^2 \leq \mathop{\mathbf{E}}_{\omega\sim\Omega} g_\omega(v,\beta)^2,$$

for each $v, \beta$. Since $\mathbf{E}_{\omega\sim\Omega} \|g_\omega(v,\cdot)\|^2 = 1$ for each $v \in V$, this means that $\mathbf{E}_{\omega,t} \|g_\omega^{(t)}(v,\cdot)\|^2$ is between $\frac{1}{8k}$ and 1. The upper bound is necessary to show that $(g_\omega^{(t)}, \Omega')$ is a valid solution to the projection games relaxation, and the two bounds prove the last part of the lemma. In addition, the lower bound shows that $\mathbf{E}_{\omega,t} \|g_\omega^{(t)}\|^2 \geq \frac{1}{8k}$.

Let us calculate how well this solution does on average:

$$\begin{aligned}\mathop{\mathbf{E}}_{\omega\sim\Omega} \mathop{\mathbf{E}}_{t\sim\mu_\omega} \langle g_\omega^{(t)}, (Id - G^\top G) g_\omega^{(t)}\rangle &\leq \left(\frac{1}{2} + \frac{\sqrt{2}}{\sqrt{1-\sigma_k^2}}\right) \mathop{\mathbf{E}}_{\omega\sim\Omega} \langle g_\omega, (Id - G^\top G) g_\omega\rangle \\ &\leq \left(1 + \frac{2\sqrt{2}}{\sqrt{1-\sigma_k^2}}\right)\eta. \qquad \text{(by Proposition 3)}\end{aligned}$$

Finally, using the bound $1 + 2\sqrt{2} \leq 4$ and rearranging slightly gives the inequality in the lemma. (We note that $\Sigma'$ is not a finite set, but as in [DS13] this is not an issue.)

With $(g'_\omega, \Omega')$ in hand, the rest of the proof will closely follow the proof of Lemma 5.5 from [DS13]. The main difference is that $\mathbf{E}_{\omega\sim\Omega'}\|Gg'_\omega\|^2$ is guaranteed to be large only in relation to $\mathbf{E}_{\omega\sim\Omega'}\|g'_\omega\|^2$, and not just in isolation. However, we have the guarantee that $g'_\omega$ assigns each $v \in V$ a value a nonnegligible fraction of the time (i.e. with measure at least $\frac{1}{8k}$). This fact will allow us to extract a good assignment from $(g'_\omega, \Omega')$.

**Lemma 4.** *There exists an assignment* $A : V \times \Sigma \to \{0, 1\}$ *such that*

$$\|GA\|^2 \geq 1 - \frac{64k\eta}{\sqrt{1-\sigma_k^2}}.$$

*Proof.* Set $\nu = 4\eta/\sqrt{1-\sigma_k^2}$. Let $\overline{\Omega}$ be $\Omega'$ rescaled to be a probability distribution. In other words, set $s := \mathbf{E}_{\Omega'} 1$, and set $\overline{\Omega} = \frac{\Omega'}{s}$. Then

- $\mathbf{E}_{\omega\sim\Omega'}\|Gg'_\omega\|^2 \geq s \cdot \mathbf{E}_{\omega\sim\overline{\Omega}}\|g'_\omega\|^2 - \nu$, and
- $\mathbf{E}_{\omega\sim\overline{\Omega}}\|g'_\omega(v, \cdot)\|^2 \geq \frac{1}{8ks}$, for all $v$.

To produce the assignment $A$ guaranteed by the lemma, we have to somehow combine the various $g'_\omega$s, each of which may only assign a label to a small fraction of the vertices in $V$, into a single global assignment for all of the vertices in $V$. To do this, we use the *correlated sampling* approach of [BHH+08]. Let $A : V \times \Sigma \to \{0, 1\}$ be the assignment generated as follows:

1. Repeatedly sample $\omega \sim \overline{\Omega}$.
2. The first time that $g'_\omega(v, \beta) \neq 0$ for some $\beta \in \Sigma$, set $A(v, \beta) = 1$.
3. Stop when all values of $A$ have been determined.

Note that once $A(v, \beta)$ is set to 1 for *some* $\beta \in \Sigma$, then $A(v, \beta')$ will remain 0 for the rest of the procedure, for any $\beta' \neq \beta$. Thus, $A$ is a valid assignment. We will show that this $A$ matches the performance guaranteed by the lemma statement, at least in expectation. To do so, we will look at the following expression for $\|GA\|^2$:

$$\|GA\|^2 = \mathop{\mathbf{E}}_{u\in U}\mathop{\mathbf{E}}_{v_1,v_2\sim u}\langle G_{u\leftarrow v_1}A(v_1, \cdot), G_{u\leftarrow v_2}A(v_2, \cdot)\rangle.$$

Note that for a fixed $u, v_1, v_2$, the inner product $\langle G_{u\leftarrow v_1}A(v_1, \cdot), G_{u\leftarrow v_2}A(v_2, \cdot)\rangle$ is 0/1-valued. Thus, we can view this as a constraint on the assignment $A$, and we will say that $A$ *satisfies* the constraint $C = (u, v_1, v_2)$ if this inner product evaluates to 1. We will calculate $\|GA\|^2$ by analyzing the probability it satisfies each constraint.

Fix a constraint $C = (u, v_1, v_2)$. We will underestimate the probability that $A$ satisfies $C$ by counting it only when $A$ gets its assignments for $v_1$ and $v_2$ from the same $\omega \sim \overline{\Omega}$. To do this, say that $g'_\omega$ *satisfies* $C$ if $g'_\omega$ assigns values to both $v_1$ and $v_2$, and these values satisfy $C$. Then

$$\mathbf{Pr}[A \text{ satisfies } C] \geq \frac{\mathbf{Pr}_{\omega\sim\overline{\Omega}}[g'_\omega \text{ satisfies } C]}{\mathbf{Pr}_{\omega\sim\overline{\Omega}}[g'_\omega \text{ assigns at least one of } v_1 \text{ and } v_2]}.$$

Set $p_1 := \mathbf{E}_{\overline{\Omega}}\|g'_\omega(v_1, \cdot)\|^2$ and $p_2 := \mathbf{E}_{\overline{\Omega}}\|g'_\omega(v_2, \cdot)\|^2$. Note that $\mathbf{Pr}_\omega[g'_\omega \text{ assigns } v_1] = p_1$, and $\mathbf{Pr}_\omega[g'_\omega \text{ assigns } v_2] = p_2$. Thus, the denominator could potentially be any number between $\max\{p_1, p_2\}$ and $p_1 + p_2$. However, the next lemma, which is Lemma 5.8 in [DS13], shows that in fact we can take the denominator to be $\frac{1}{2}(p_1 + p_2)$ with little cost:

**Lemma 5.** *For a constraint* $C = (u, v_1, v_2)$*, set*

$$\gamma_C := 1 - \left( \frac{\mathbf{Pr}_{\overline{\Omega}}[g'_\omega \text{ satisfies } C]}{\frac{1}{2}\left(\mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_1,\cdot)\|^2 + \mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_2,\cdot)\|^2\right)} \right).$$

*Then* $\mathbf{Pr}[A \text{ satisfies } C] \geq (1-\gamma_C)(1+\gamma_C)$.

Set $\psi(z) = (1-z)/(1+z)$. In addition, let $C \sim G$ denote the natural distribution on constraints, i.e. the distribution which selects $u \in U$ uniformly at random, selects $v_1, v_2 \sim u$ independently and uniformly at random, and outputs $(u, v_1, v_2)$. Now, we have

$$\mathbf{E}\, \|GA\|^2 = \mathop{\mathbf{E}}_{C \sim G} \mathbf{Pr}[A \text{ satisfies } C] \geq \mathop{\mathbf{E}}_{C} \psi(\gamma_C) \geq \psi\left(\mathop{\mathbf{E}}_{C} \gamma_C\right), \tag{4}$$

where the final step uses the convexity of $\psi$. To bound $\mathbf{E}_C\, \gamma_C$,

$$1 - \mathop{\mathbf{E}}_{C} \gamma_C = \mathop{\mathbf{E}}_{C} \left( \frac{2}{\mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_1,\cdot)\|^2 + \mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_2,\cdot)\|^2} \right) \mathop{\mathbf{Pr}}_{\overline{\Omega}}[g'_\omega \text{ satisfies } C]$$

Set $X_C = \frac{1}{2}\left(\mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_1,\cdot)\|^2 + \mathbf{E}_{\overline{\Omega}} \|g'_\omega(v_2,\cdot)\|^2\right)$ and $Y_C = \mathbf{Pr}_{\overline{\Omega}}[g'_\omega \text{ satisfies } C]$. Then we note that

- $X_C \geq \frac{1}{8ks}$, and
- $\mathbf{E}_C\, Y_C \geq \mathbf{E}_C\, X_C - \frac{\nu}{s}$.

Now, we calculate

$$\begin{aligned} \mathop{\mathbf{E}}_{C} \gamma_C = \left(1 - \mathop{\mathbf{E}}_{C} \frac{Y_C}{X_C}\right) &= \left(\mathop{\mathbf{E}}_{C} \frac{X_C - Y_C}{X_C}\right) \\ &\leq 8ks \left(\mathop{\mathbf{E}}_{C} X_C - Y_C\right) && \left(\text{because } X_C \geq \tfrac{1}{8ks} \text{ always}\right) \\ &\leq 8k\nu. \end{aligned}$$

By Equation (4), $\mathbf{E}\, \|GA\|^2 \geq (1 - 8k\cdot\nu)/(1 + 8k\cdot\nu) \geq 1 - 16k\cdot\nu$. By the probabilistic method, an assignment must therefore exist which achieves this value. Substituting $\nu = 4\eta/\sqrt{1-\sigma_k^2}$ yields the lemma.

Finally, we can show an assignment $(f, A)$ for which $\langle f, GA\rangle$ is large. This is because

$$\|GA\|^2 = \langle GA, GA\rangle \leq \max_f \langle f, GA\rangle \leq \mathrm{val}(G),$$

where the max is taken over assignments to $U$. Thus, $\|GA\|^2$ is a lower bound on the value of $G$, and we are done.

## 4 Parallel repetition for expanders

In this section, we give a simple proof of Theorem 7 in the expanding games case, albeit with a worse dependence on $\sigma_2$. Our proof follows from combining the following theorem with Lemma 1.

**Theorem 8.** *Let* $G$ *be a projection game over vertex set* $U \cup V$*. Let* $\sigma_2$ *be the second singular value of* $G$*'s constraint graph* $H$*. If* $\mathcal{S} = (f, g, \Omega)$ *is a solution to the relaxation with* $\mathrm{val}_+(\mathcal{S}) \geq 1 - \eta$*, then*

$$\mathrm{val}(G) \geq 1 - \left(\frac{16\eta}{1-\sigma_2^2}\right).$$

*Proof.* Lemma 3.1 of [DS13] shows that in this case, there exists a single fractional assignment $g : V \times \Sigma \to \mathbb{R}^{\geq 0}$ such that $\|Gg\|^2 \geq 1-\eta$ and $\|g\| = 1$. Write $g(v,i) = h(v) \cdot I(v,i)$, as in Section 2. Recall from Equation (3) that we can write $H$ as

$$H = \sum_{i=1}^{d} \sigma_i \cdot l_i \cdot r_i^\top, \tag{5}$$

where $l_1$ and $r_1$ are the constantly-one vectors. Then $\|Hh\|^2 \geq 1-\eta$ and $\|h\|^2 = 1$. The following proposition shows that $h$ itself must be mostly constant.

**Proposition 4.** *Write $h = a_1 r_1 + \ldots + a_d r_d + o$, where $o$ is orthogonal to $r_1, \ldots, r_d$. Then*

$$a_1^2 \geq 1 - \left(\frac{\eta}{1-\sigma_2^2}\right).$$

*Proof.* We can write $Hh = \sum_{i=1}^{d} \sigma_i a_i l_i$, and so

$$\begin{aligned} \|Hh\|^2 = \sum_{i=1}^{d} \sigma_i^2 a_i^2 \; &\leq \; a_1^2 + \sigma_2^2 \cdot \sum_{i}^{r} a_i^2 \\ &= \; a_1^2 + \sigma_2^2 \cdot (1-a_1^2) \; = \; (1-\sigma_2^2) \cdot a_1^2 + \sigma_2^2. \end{aligned}$$

Finally, we know that $\|Hh\|^2 \geq (1-\eta)$. Putting everything together,

$$a_1^2 \geq \left(\frac{1-\eta-\sigma_2^2}{1-\sigma_2^2}\right),$$

giving us the proposition.

Because $g$ is a good fractional assignment and Proposition 4 shows that it must be near-constant, it is reasonable to suppose that in fact the function $I$ is a good 0/1-assignment. With this in mind, it is our goal to show that $\|GI\|^2$ is large given that $\|Gg\|^2$ is large. To this end, define $h' := h - a_1 r_1$, and rewrite $\|Gg\|^2$ as follows:

$$\begin{aligned} \langle Gg, Gg \rangle &= \langle G \cdot hI, G \cdot hI \rangle \\ &= \langle G(a_1 r_1 + h')I, G(a_1 r_1 + h')I \rangle \\ &= a_1^2 \langle G \cdot r_1 I, G \cdot r_1 I \rangle + 2a_1 \langle G \cdot r_1 I, G \cdot h'I \rangle + \langle G \cdot h'I, G \cdot h'I \rangle. \end{aligned} \tag{6}$$

Now, $r_1$ is the constantly-one function, so $r_1 I = I$. Next,

$$\langle G \cdot h'I, G \cdot h'I \rangle \leq \|h'I\|^2 = \|h'\|^2 \leq \left(\frac{\eta}{1-\sigma_2^2}\right),$$

where the first inequality uses the fact that $G$ is norm-reducing. Applying these to Equation (6), along with the fact that $\|Gg\|^2 \geq 1-\eta$, yields

$$a_1^2 \langle GI, GI \rangle + 2a_1 \langle GI, G \cdot h'I \rangle \geq 1 - \eta - \left(\frac{\eta}{1-\sigma_2^2}\right).$$

If $\langle GI, G \cdot h'I \rangle$ is negative, then $\langle G \cdot I, G \cdot I \rangle \geq 1 - 2\eta$, and we are done. Otherwise, we may upper bound both $a_1^2$ and $a_1$ with 1, giving

$$\langle GI, GI \rangle + 2\langle GI, G \cdot h'I \rangle \geq 1 - \eta - \left(\frac{\eta}{1-\sigma_2^2}\right). \tag{7}$$

It is now our goal to show that $\langle GI, G \cdot h'I \rangle$ must be $O(\eta)$. In fact, we will show the following lemma.

**Lemma 6.** $\langle GI, G \cdot h'I \rangle \leq \sqrt{\frac{\eta \cdot (1-\|GI\|^2)}{1-\sigma_2^2}}$.

*Proof.* For the proof of this lemma, it will be convenient to define the operator $(GI)$ which acts on functions $\nu : V \to \mathbb{R}$ as follows: $(GI)\nu := G(I \cdot \nu)$. For any such function $\nu$, $I \cdot \nu$ is a fractional assignment. As a result, $(GI)$ is norm-reducing:

$$\|(GI)\nu\| = \|G(I \cdot \nu)\| \leq \|I \cdot \nu\| = \|\nu\|.$$

Here the inequality follows from the fact that $G$ is norm-reducing on fractional assignments. In particular, $(GI)$'s singular values lie in the interval $[0, 1]$.

Set $N_V = |V|$. Let $v_1, \ldots, v_{N_V}$ be the eigenvectors of $(GI)^\top(GI)$ with eigenvalues $\lambda_1, \ldots, \lambda_{N_V}$. Then we can write $(GI)^\top(GI)$ as follows:

$$(GI)^\top(GI) = \sum_{i=1}^{N_V} \lambda_i \cdot v_i \cdot v_i^\top.$$

Because $(GI)$ is norm-reducing, we know that $\lambda_i \in [0, 1]$ for all $i$.

Let $\mathbf{1} : V \to \mathbb{R}$ denote the all-ones vector (i.e., $\mathbf{1}(v) = 1$ for all $v$). As $h'$ is orthogonal to $\mathbf{1}$, we have that $\langle \mathbf{1}, h' \rangle = 0$. Thus,

$$\begin{aligned} 0 = \langle \mathbf{1}, h' \rangle &= \mathbf{1}^\top \cdot Id \cdot h' \\ &= \mathbf{1}^\top \left( \sum_{i=1}^{N_V} v_i \cdot v_i^\top \right) h' \\ &= \sum_{i=1}^{N_V} \langle \mathbf{1}, v_i \rangle \cdot \langle v_i, h' \rangle. \end{aligned} \tag{8}$$

Next, the quantity we are trying to compute is

$$\begin{aligned} \langle GI, G(Ih') \rangle = \langle (GI)\mathbf{1}, (GI)h' \rangle = \mathbf{1}^\top (GI)^\top (GI) h' \\ = \mathbf{1}^\top \cdot \left( \sum_{i=1}^{N_V} \lambda_i \cdot v_i \cdot v_i^\top \right) \cdot h' = \sum_{i=1}^{N_V} \lambda_i \langle \mathbf{1}, v_i \rangle \cdot \langle v_i, h' \rangle. \end{aligned}$$

Because Equation (8) equals zero, we can subtract it from this equation and get that

$$\begin{aligned} \langle GI, G(Ih') \rangle &= \sum_{i=1}^{N_V} (\lambda_i - 1) \langle \mathbf{1}, v_i \rangle \cdot \langle v_i, h' \rangle \\ &\leq \sqrt{\sum_{i=1}^{N_V} |\lambda_i - 1| \cdot \langle \mathbf{1}, v_i \rangle^2} \cdot \sqrt{\sum_{i=1}^{N_V} |\lambda_i - 1| \cdot \langle v_i, h' \rangle^2}. \end{aligned}$$

Now, because $\lambda_i \in [0, 1]$, $|\lambda_i - 1| = 1 - \lambda_i$. As

$$\sum_{i=1}^{N_V} (1 - \lambda_i) \cdot \langle \mathbf{1}, v_i \rangle^2 = \|\mathbf{1}\|^2 - \|(GI)\mathbf{1}\|^2,$$

and similarly for $h'$, we have that

$$\begin{aligned}\langle GI, G(Ih')\rangle &\leq \sqrt{\|\mathbf{1}\|^2 - \|(GI)\mathbf{1}\|^2} \cdot \sqrt{\|h'\|^2 - \|(GI)h'\|^2} \\ &\leq \sqrt{1 - \|GI\|^2} \cdot \|h'\| \\ &\leq \sqrt{\frac{\eta \cdot (1 - \|GI\|^2)}{1 - \sigma_2^2}},\end{aligned}$$

which completes the proof.

Now we can show the main result. Combining Lemma 6 and Equation 7, we see that

$$1 - \eta - \left(\frac{\eta}{1 - \sigma_2^2}\right) \leq \|GI\|^2 + 2\sqrt{\frac{\eta \cdot (1 - \|GI\|^2)}{1 - \sigma_2^2}}.$$

Rearranging this,

$$(1 - \|GI\|^2) \leq 2\sqrt{\frac{\eta}{1 - \sigma_2^2}} \cdot \sqrt{1 - \|GI\|^2} + \eta + \left(\frac{\eta}{1 - \sigma_2^2}\right).$$

Now, either the first term on the right-hand side is larger than the next two terms or the next two terms are larger than the first term. In the first case,

$$(1 - \|GI\|^2) \leq 4\sqrt{\frac{\eta}{1 - \sigma_2^2}} \cdot \sqrt{1 - \|GI\|^2}.$$

Dividing both sides by $\sqrt{1 - \|GI\|^2}$ and squaring, we get that

$$(1 - \|GI\|^2) \leq 16\left(\frac{\eta}{1 - \sigma_2^2}\right).$$

In the second case,

$$(1 - \|GI\|^2) \leq 2\left(\eta + \left(\frac{\eta}{1 - \sigma_2^2}\right)\right).$$

In both cases, the bound is at most $(1 - \|GI\|^2) \leq 16\eta/(1 - \sigma_2^2)$. Thus, $\|GI\|^2 \geq 1 - 16\eta/(1 - \sigma_2^2)$. As a result, there is an assignment $f$ to the $U$ vertices such that $\langle f, GI\rangle \geq 1 - 16\eta/(1 - \sigma_2^2)$, and we are done.

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## A Proof of Lemma 3

**Lemma 7 (Lemma 3 restated).** *Let $g : V \times \Sigma \to \mathbb{R}^{\geq 0}$ be a fractional assignment for $G$. Then there exists a measure $\mu : [0, \infty) \to \mathbb{R}^{\geq 0}$ and a 0/1-assignment $g^{(t)} : V \times \Sigma \to \{0, 1\}$, for each $t \geq 0$, such that*

$$\mathop{\mathbf{E}}_{t\sim\mu} \langle g^{(t)}, (Id - G^\top G) g^{(t)} \rangle \leq \left( \frac{1}{2} + \frac{\sqrt{2}}{\sqrt{1 - \sigma_k^2}} \right) \langle g, (Id - G^\top G) g \rangle,$$

*Furthermore, $\frac{1}{8k} g(v, \beta)^2 \leq \mathbf{E}_{t\sim\mu}\, g^{(t)}(v, \beta)^2 \leq g(v, \beta)^2$, for each $v \in V, \beta \in \Sigma$.*

*Proof.* Our proof of this lemma follows the (first) proof of Theorem 1.2 from [KLL+13]. We will first show that there exists a $(2k + 1)$-step function which approximates $g$ well, and then we will use this to show how to round $g$ into a good 0/1-assignment. We can accomplish the first step by essentially using Lemma 3.1 from [KLL+13] as a black box, but the second step requires us to tailor the proof of Proposition 3.2 from [KLL+13] to our particular setting. The reason for this is as follows: when we compute norms, we use the uniform measure over the vertices. However, in [KLL+13], the norms give each vertex a weight depending on their degree. In our first step, we will be using the graph $H^\top H$, which is regular, so these two norms coincide. In our second step, however, we will be using the graph $G^\top G$, which is not necessarily regular, and so we will have to perform some extra work.

Set $\rho \geq 0$ so that $\|Gg\|^2 = (1-\rho)\|g\|^2$. Write $g(v,i) = h(v) \cdot I(v,i)$, as in Section 2. We begin with the following proposition.

**Proposition 5.** *There is a $(2k+1)$-step approximation of $h$, called $\widetilde{h}$, such that*

$$\|h - \widetilde{h}\|^2 \leq \left(\frac{4\rho}{1-\sigma_k^2}\right) \cdot \|h\|^2.$$

*Proof.* From Section 2, we know that $\|Hh\|^2 \geq \|Gg\|^2$ and $\|h\|^2 = \|g\|^2$. As a result, $\|Hh\|^2 \geq (1-\rho)\|h\|^2$. Another way of writing $\|Hh\|^2 = \langle Hh, Hh\rangle$ is $\langle h, H^\top H h\rangle$. Using Equation (3), $H^\top H$ is equal to

$$H^\top H = \sum_{i=1}^{d} \sigma_i^2 \cdot r_i \cdot r_i^\top.$$

$H^\top H$ is the adjacency matrix corresponding to the graph on $V$ whose edges are distributed as follows: first pick $u \in U$ uniformly at random, and output the pair $(v_1, v_2)$, where $v_1$ and $v_2$ are independent and uniformly random neighbors of $u$. Because the constraint graph of $G$ is biregular, the graph $H^\top H$ corresponds to is regular.

The Laplacian of this graph is $Id - H^\top H$, and its eigenvalues are $0 = 1 - \sigma_1^2 \leq \ldots \leq 1 - \sigma_d^2$, in addition to the eigenvalue 1 (with some multiplicity). As

$$\langle h, (Id - H^\top H)h\rangle = \langle h, Id \cdot h\rangle - \langle h, H^\top H h\rangle \leq \rho\|h\|^2,$$

we can apply Lemma 3.1 from [KLL+13] to get a $(2k+1)$-step approximation of $h$, called $\widetilde{h}$, such that

$$\|h - \widetilde{h}\|^2 \leq \left(\frac{4\rho}{1-\sigma_k^2}\right) \cdot \|h\|^2.$$

As mentioned above, we are able to apply their lemma because $H^\top H$ is a regular graph. We note that the graph $H^\top H$ has self-loops, and while their Lemma 3.1 does not explicitly mention this, the proof goes through equally well in this case.

We would now like to apply Proposition 3.2 from [KLL+13] to show how to round $g$ into a good 0/1-assignment. Unfortunately, as mentioned above, $G$ is not necessarily biregular (as $G$ is a projection game, the vertices on the $V$ side have the same degree, but the $U$ vertices may not), and so $G^\top G$ is not necessarily regular. However, $H^\top H$ is regular, and this allows us to follow roughly the same proof as [KLL+13].

**Proposition 6.** *There exists a measure $\mu : [0,\infty) \to \mathbb{R}^{\geq 0}$ and a 0/1-assignment $g^{(t)} : V \times \Sigma \to \mathbb{R}^{\geq 0}$, for each $t \geq 0$, such that*

$$\mathop{\mathbf{E}}_{t\sim\mu} \langle g^{(t)}, (Id - G^\top G) g^{(t)}\rangle \leq \frac{1}{2}\langle g, (Id - G^\top G) g\rangle + \frac{1}{\sqrt{2}}\sqrt{\langle g, (Id - G^\top G) g\rangle} \cdot \|h - \widetilde{h}\|$$

*Proof.* To begin, we will write $\langle g, G^\top G g\rangle$ in a way that separates the underlying constraint graph $H$ from the constraints $\pi_{u,v}$ on the edges. By Equation (2),

$$\langle g, G^\top G g\rangle = \mathop{\mathbf{E}}_{u\in U} \mathop{\mathbf{E}}_{v_1,v_2\sim u} \langle G_{u\leftarrow v_1} g(v_1,\cdot), G_{u\leftarrow v_2} g(v_2,\cdot)\rangle.$$

Note that as $g$ is a fractional assignment, for any neighbors $u$ and $v$,

$$\|G_{u\leftarrow v}g(v,\cdot)\|^2 = \|g(v,\cdot)\|^2.$$

As a result,

$$\begin{aligned}\langle g, Id\cdot g\rangle &= \mathop{\mathbf{E}}_{u\in U}\mathop{\mathbf{E}}_{v_1,v_2\sim u}\left(\frac{\|g(v_1,\cdot)\|^2+\|g(v_2,\cdot)\|^2}{2}\right)\\ &= \mathop{\mathbf{E}}_{u\in U}\mathop{\mathbf{E}}_{v_1,v_2\sim u}\left(\frac{\|G_{u\leftarrow v_1}g(v_1,\cdot)\|^2+\|G_{u\leftarrow v_2}g(v_2,\cdot)\|^2}{2}\right).\end{aligned}$$

This gives the following familiar-ish expression for the Laplacian of the game:

$$\langle g,(Id-G^\top G)g\rangle = \mathop{\mathbf{E}}_{u\in U}\mathop{\mathbf{E}}_{v_1,v_2\sim u}\left(\frac{\|G_{u\leftarrow v_1}g(v_1,\cdot)-G_{u\leftarrow v_2}g(v_2,\cdot)\|^2}{2}\right).$$

As $\|Gg\|^2=(1-\rho)\|g\|^2$, this Laplacian equals $\rho\|g\|^2$.

Now, let us define the randomized rounding procedure for $g$. Given $t\geq 0$, first define the threshold function $h^{(t)}(v)=\mathbf{1}[h(v)\geq t]$. Let $M$ be the maximum value of $h$, and let $\mu:[0,M]\to\mathbb{R}$ be the measure defined in the proof of Proposition 3.2 of [KLL+13]. Then by Claim 3.3 of [KLL+13],

$$\mathop{\mathbf{E}}_{t\sim\mu}[h^{(t)}(v)^2]\geq\frac{1}{8k}h(v)^2.$$

Averaging this over all $v\in V$, $\mathbf{E}_{t\sim\mu}\|h^{(t)}\|^2\geq\frac{1}{8k}\|h\|^2$. In addition, though this is not stated anywhere in [KLL+13], it is easy to see that

$$\mathop{\mathbf{E}}_{t\sim\mu}[h^{(t)}(v)^2]\leq\frac{1}{2}h(v)^2\leq h(v)^2$$

as well.[3] Using this, we define the threshold function for $g$ as $g^{(t)}=h^{(t)}\cdot I$.

Next, we translate Claim 3.4 from [KLL+13] into our language:

*Claim.* Let $u\in U$ and $v_1,v_2\in V$ be adjacent to $u$. Write $g_1(\alpha)=(G_{u\leftarrow v_1}g(v_1,\cdot))(\alpha)$, $g_2(\alpha)=(G_{u\leftarrow v_2}g(v_2,\cdot))(\alpha)$, and define $g_1^{(t)}$ and $g_2^{(t)}$ analogously. Then

$$\mathop{\mathbf{E}}_{t\sim\mu}\left\|g_1^{(t)}-g_2^{(t)}\right\|^2\leq\frac{1}{2}\|g_1-g_2\|\cdot\left(|h(v_1)-\widetilde{h}(v_1)|+|h(v_2)-\widetilde{h}(v_2)|+\|g_1-g_2\|\right).$$

*Proof.* In the case when $v_1=v_2$, both sides of inequality are zero, and so the statement is true. Otherwise, let $\beta_1,\beta_2\in\Sigma$ be the labels assigned to $v_1$ and $v_2$ by $g$, meaning that $I(v_1,\beta_1)=I(v_2,\beta_2)=1$. Let us consider the following two cases.

*Case $\pi_{uv_1}(\beta_1)\neq\pi_{uv_2}(\beta_2)$:* In this case, we will show simply that

$$\mathop{\mathbf{E}}_{t\sim\mu}\left\|g_1^{(t)}-g_2^{(t)}\right\|^2\leq\frac{1}{2}\|g_1-g_2\|^2.$$

Because $\pi_{uv_1}(\beta_1)\neq\pi_{uv_2}(\beta_2)$, the left-hand side is $\mathbf{E}_{t\sim\mu}\|g^{(t)}(v_1,\cdot)\|^2+\|g^{(t)}(v_2,\cdot)\|^2$, and the right-hand side is $\frac{1}{2}\left(\|g(v_1,\cdot)\|^2+\|g(v_2,\cdot)\|^2\right)$. The inequality now follows, as

$$\mathop{\mathbf{E}}_{t\sim\mu}\|g^{(t)}(v,\cdot)\|^2\leq\frac{1}{2}\|g(v,\cdot)\|^2,$$

for any $v\in V$.

[3] This follows because $\int_0^t x\,\mathrm{d}x=t^2/2$.

*Case* $\pi_{uv_1}(\beta_1) = \pi_{uv_2}(\beta_2)$*:* This case follows directly from the proof of Claim 3.4 in [KLL+13], completing the proof.

With this in place, the rest of the proof follows the proof of Proposition 3.2 in [KLL+13], and we get that

$$\mathop{\mathbf{E}}_{t\sim\mu}\langle g^{(t)}, (Id - G^\top G)g^{(t)}\rangle \leq \frac{1}{2}\langle g, (Id - G^\top G)g\rangle + \frac{1}{\sqrt{2}}\sqrt{\langle g, (Id - G^\top G)g\rangle} \cdot \|h - \widetilde{h}\|,$$

the bound we were looking for.

Now, by combining Propositions 5 and 6 we see that

$$\begin{aligned}\mathop{\mathbf{E}}_{t\sim\mu}\langle g^{(t)}, (Id - G^\top G)g^{(t)}\rangle &\leq \frac{1}{2}\rho\|g\|^2 + \frac{\sqrt{2}}{\sqrt{1-\sigma_k^2}}\rho\|g\|^2 \\ &= \left(\frac{1}{2} + \frac{\sqrt{2}}{\sqrt{1-\sigma_k^2}}\right)\rho\|g\|^2.\end{aligned}$$

As $\rho\|g\|^2 = \langle g, (Id - G^\top G)g\rangle$, this is as guaranteed by the lemma. The bounds on $\mathbf{E}_{t\sim\mu}\, g^{(t)}(v,\beta)$ follow from Proposition 6.