Higher-order Functions

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Introduction

(f1) :: Int → Int

A function is called higher-order if it takes a function as an argument or returns a function as a result.

```
twice     :: (a → a) → a → a
twice f x = f (f x)
```

twice (f1) ::

(Num a) =>

a → a

twice is ***higher-order*** because it takes a function as its first argument.

Why Are They Useful?

- Common programming idioms can be encoded as functions within the language itself.

$f \$ f \equiv \text{twice } f$

- Domain specific languages can be defined as collections of higher-order functions.

- Algebraic properties of higher-order functions can be used to reason about programs.

The Map Function

map f [a,b,c,d]

→ [f a, f b, f c, f d]

The higher-order library function called map applies a function to every element of a list.

```
map :: (a → b) → [a] → [b]
```

For example:

```
> map (+1) [1,3,5,7]

[2,4,6,8]
```

The map function can be defined in a particularly simple manner using a list comprehension:

```
map f xs = [f x | x ← xs]
```

Alternatively, for the purposes of proofs, the map function can also be defined using recursion:

```
map f []     = []
map f (x:xs) = f x : map f xs
```

The Filter Function

The higher-order library function filter selects every element from a list that satisfies a predicate.

```
filter :: (a → Bool) → [a] → [a]
```

For example:

```
> filter even [1..10]

[2,4,6,8,10]
```

Filter can be defined using a list comprehension:

```
filter p xs = [x | x ← xs, p x]
```

Alternatively, it can be defined using recursion:

```
filter p []         = []
filter p (x:xs)
   | p x            = x : filter p xs
   | otherwise      = filter p xs
```

Examples:

```
sum []     = 0
sum (x:xs) = x + sum xs
```

$v = 0$
$\oplus = +$

```
product []     = 1
product (x:xs) = x * product xs
```

$v = 1$
$\oplus = *$

```
and []     = True
and (x:xs) = x && and xs
```

$v = True$
$\oplus = \&\&$

The Foldr Function

A number of functions on lists can be defined using the following simple pattern of recursion:

```
f []     = v
f (x:xs) = x ⊕ f xs
```

f maps the empty list to some value v, and any non-empty list to some function ⊕ applied to its head and f of its tail.

The higher-order library function foldr (fold right) encapsulates this simple pattern of recursion, with the function ⊕ and the value v as arguments.

For example:

```
sum     = foldr (+) 0

product = foldr (*) 1

or      = foldr (||) False

and     = foldr (&&) True
```

Foldr itself can be defined using recursion:

```
foldr :: (a → b → b) → b → [a] → b
foldr f v []     = v
foldr f v (x:xs) = f x (foldr f v xs)
```

It is best to think of foldr non-recursively, as simultaneously replacing each (:) in a list by a given function, and [] by a given value.

```
  sum [1,2,3]

= foldr (+) 0 [1,2,3]

= foldr (+) 0 (1:(2:(3:[])))

= 1+(2+(3+0))

= 6
```

Replace each (:) by (+) and [] by 0.

```
  product [1,2,3]

= foldr (*) 1 [1,2,3]

= foldr (*) 1 (1:(2:(3:[])))

= 1*(2*(3*1))

= 6
```

Replace each (:) by (*) and [] by 1.

Other Foldr Examples

Even though foldr encapsulates a simple pattern of recursion, it can be used to define many more functions than might first be expected.

```
length        :: [a] → Int
length []      = 0
length (_:xs) = 1 + length xs
```

length [1,2,3]

= length (1:(2:(3:[])))

= 1+(1+(1+0))

= 3

Replace each (:) by λ _ n → 1+n; and [] by 0.

Hence, we have:

length = foldr (\ _ n → 1+n) 0

Now recall the reverse function:

```
reverse []     = []
reverse (x:xs) = reverse xs ++ [x]
```

For example:

```
  reverse [1,2,3]
= reverse (1:(2:(3:[])))
= (([] ++ [3]) ++ [2]) ++ [1]
= [3,2,1]
```

Replace each (:) by λx xs $\rightarrow$ xs ++ [x] and [] by [].

Hence, we have:

```
reverse =
   foldr (λx xs → xs ++ [x]) []
```

Finally, we note that the append function (++) has a particularly compact definition using foldr:

```
(++ ys) = foldr (:) ys
```

Replace each (:) by (:) and [] by ys.

Why Is Foldr Useful?

- Some recursive. functions on lists, such as sum, are simpler to define using foldr.

- Properties of functions defined using foldr can be proved using algebraic properties of foldr, such as fusion and the banana split rule.

- Advanced program optimizations can be simpler if foldr is used in place of explicit recursion.

Other Library Functions

The library function (.) returns the composition of two functions as a single function.

```
(.)    :: (b → c) → (a → b) → (a → c)
f . g  = λx → f (g x)
```

For example:

```
odd :: Int → Bool
odd  = not . even
```

The library function all decides if every element of a list satisfies a given predicate.

```
all      :: (a → Bool) → [a] → Bool
all p xs = and [p x | x ← xs]
```

For example:

```
> all even [2,4,6,8,10]

True
```

Dually, the library function <u>any</u> decides if at least one element of a list satisfies a predicate.

```
any      :: (a → Bool) → [a] → Bool
any p xs = or [p x | x ← xs]
```

For example:

```
> any (== ' ') "abc def"

True
```

The library function takeWhile selects elements from a list while a predicate holds of all the elements.

```
takeWhile :: (a → Bool) → [a] → [a]
takeWhile p []     = []
takeWhile p (x:xs)
   | p x           = x : takeWhile p xs
   | otherwise     = []
```

For example:

```
> takeWhile (/= ’ ’) "abc def"

"abc"
```

Dually, the function dropWhile removes elements while a predicate holds of all the elements.

```
dropWhile :: (a → Bool) → [a] → [a]
dropWhile p []      = []
dropWhile p (x:xs)
   | p x           = dropWhile p xs
   | otherwise     = x:xs
```

For example:

```
> dropWhile (== ’ ’) "   abc"

"abc"
```

Exercises

(1) What are higher-order functions that return functions as results better known as?

(2) Express the comprehension [f x | x $\leftarrow$ xs, p x] using the functions map and filter.

(3) Redefine map f and filter p using foldr.