More support for symbolic disintegration

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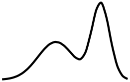
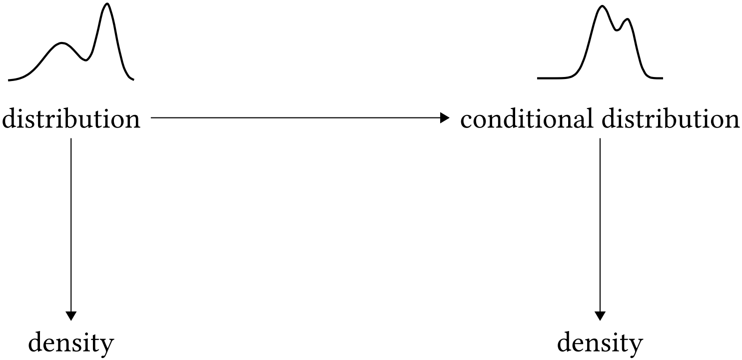
Indiana University

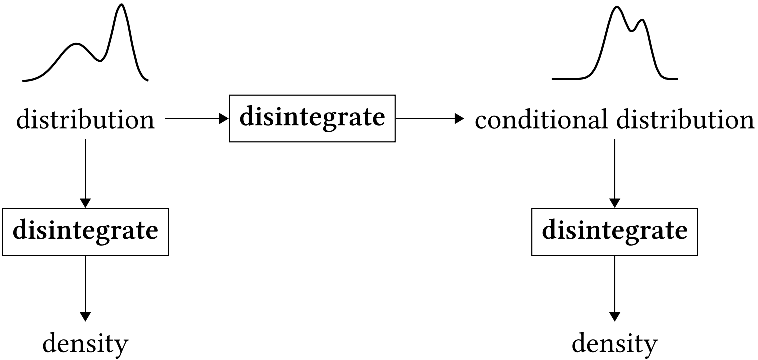
PPS, January 9 2018

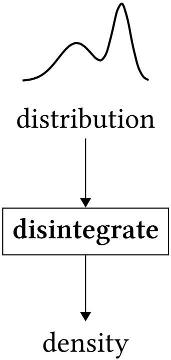


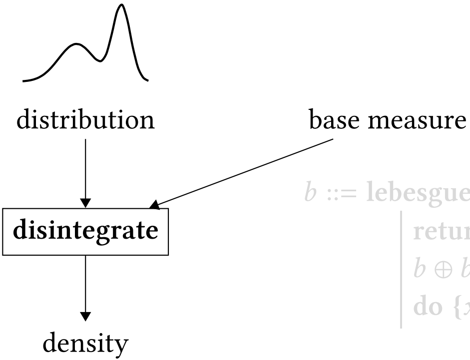
distribution


distribution
conditional distribution

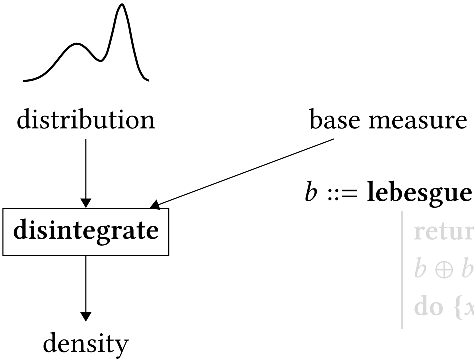

distribution
conditional distribution
density
density


distribution
disintegrate
conditional distribution
disintegrate
disintegrate
density
density


distribution
disintegrate
density



$$
\begin{aligned}
b ::=\ & \textbf{lebesgue} \\
\mid\ & \textbf{return}\ e \\
\mid\ & b \oplus b \\
\mid\ & \textbf{do}\ \{x \leftsquigarrow b;\ y \leftsquigarrow b;\ \textbf{return}\ (x\,,\,y)\}
\end{aligned}
$$



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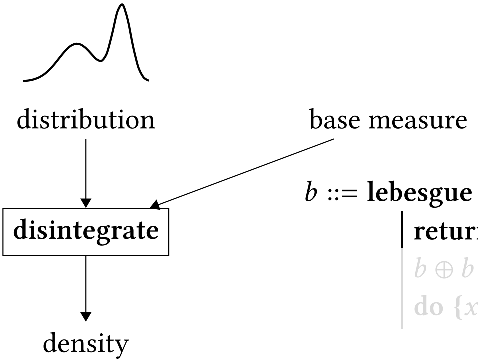
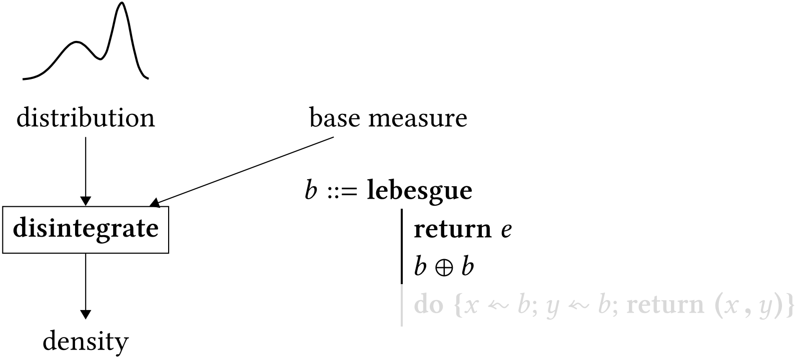
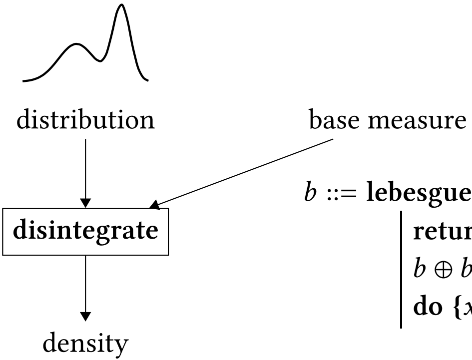


b ::= **lebesgue**
 | **return** e
 | $b \oplus b$
 | **do** $\{x \leftsquigarrow b; y \leftsquigarrow b;$ **return** $(x, y)\}$


distribution
base measure
disintegrate
density
b ::= lebesgue
| return e
| b ⊕ b
| do {x ↜ b; y ↜ b; return (x , y)}



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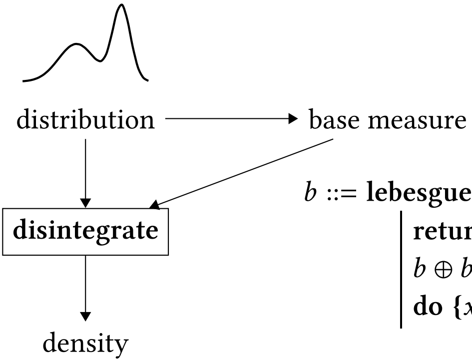

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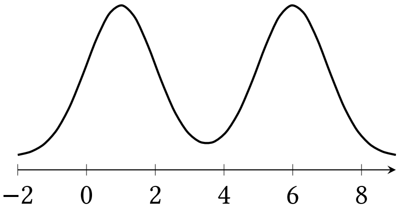
Comparing hypotheses



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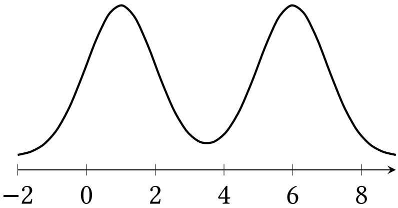

$\mathcal{N}(1,1)$, density $d_1 = \lambda x.\frac{e^{-(x-1)^2/2}}{\sqrt{2\pi}}$

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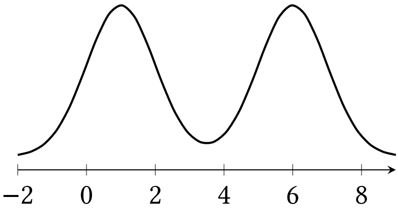

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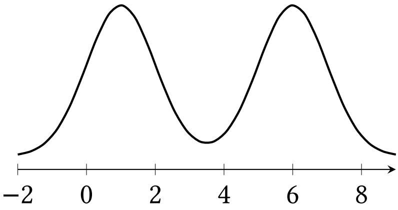

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ν has a density with respect to μ, i.e, $\nu <: \mu$
$\Leftrightarrow \exists \Delta.\, \nu\ f = \mu\ (\Delta \cdot f)$

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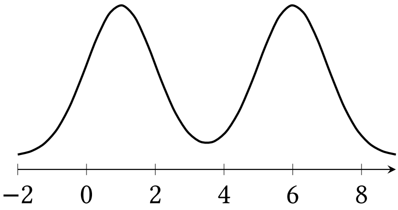

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Special case:
If $\nu = \mathcal{N}(1,1)$,
and $\mu = \Lambda$ (the Lebesgue measure),
then $\Delta = d_1$

Integrator semantics of core Hakaru

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$$[\![\mathbb{M}\ \alpha]\!] = \overbrace{([\![\alpha]\!] \to \mathbb{R})}^{\text{integrand}} \to \mathbb{R}$$

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\end{aligned}
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Density specification

$$
\begin{aligned}
&[\![\nu]\!] = \lambda f.\ [\![\mu]\!]\ (\Delta \cdot f) \\
&\nu = \textbf{do}\ \{t \leftsquigarrow \mu; \textbf{factor}\ (\Delta\, t); \textbf{return}\ t\}
\end{aligned}
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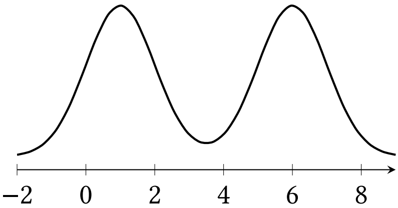
Notation

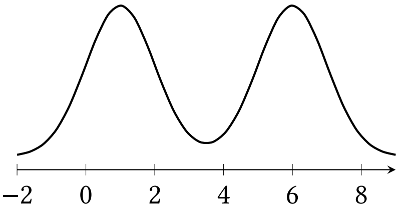
$$\nu = \mu \otimes \Delta$$

Back to GMM



(**normal** 1 1) $\oplus$ (**normal** 6 1)

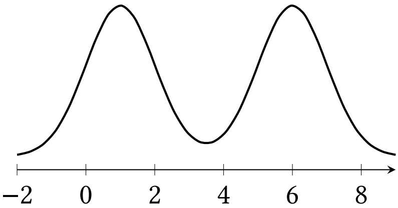
Back to GMM



(**normal** 1 1) ⊕ (**normal** 6 1)

$$\left[\!\left[\begin{array}{l} (\mathbf{normal}\ 1\ 1) \\ \oplus\ (\mathbf{normal}\ 6\ 1) \end{array} \right]\!\right]$$

Back to GMM



$$(\textbf{normal}\ 1\ 1) \oplus (\textbf{normal}\ 6\ 1)$$

$$\left[\!\left[\begin{array}{l} (\textbf{normal}\ 1\ 1) \\ \oplus\ (\textbf{normal}\ 6\ 1) \end{array} \right]\!\right] = \lambda f.\ \int_{-\infty}^{\infty} d_1\ x \cdot f\ x \cdot dx + \int_{-\infty}^{\infty} d_2\ x \cdot f\ x \cdot dx$$

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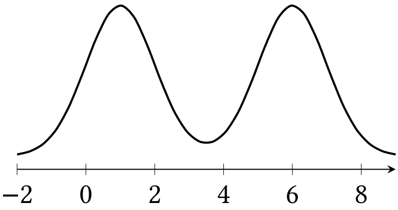

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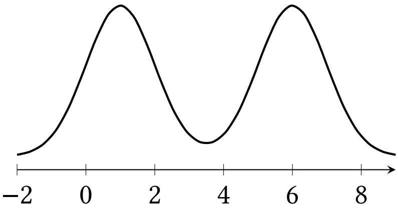
$$= \lambda f.\ [\![\textbf{lebesgue} \otimes d_1]\!]\ f + [\![\textbf{lebesgue} \otimes d_2]\!]\ f$$

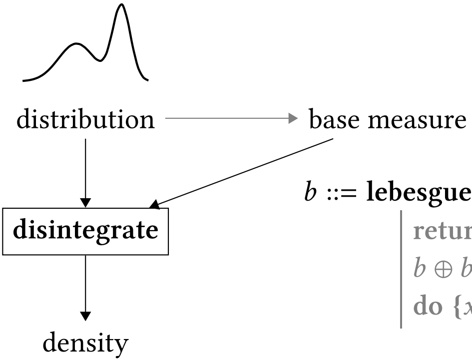
Back to GMM



$$(\textbf{normal}\ 1\ 1) \oplus (\textbf{normal}\ 6\ 1)$$

$$\begin{aligned}
\left[\!\left[\begin{array}{l}(\textbf{normal}\ 1\ 1)\\ \oplus\ (\textbf{normal}\ 6\ 1)\end{array}\right]\!\right] &= \lambda f.\ \int_{-\infty}^{\infty} d_1\ x \cdot f\ x \cdot dx + \int_{-\infty}^{\infty} d_2\ x \cdot f\ x \cdot dx \\
&= \lambda f.\ [\![\textbf{lebesgue} \otimes d_1]\!]\ f + [\![\textbf{lebesgue} \otimes d_2]\!]\ f \\
&= [\![\textbf{lebesgue} \otimes (d_1 + d_2)]\!]
\end{aligned}$$

Agenda



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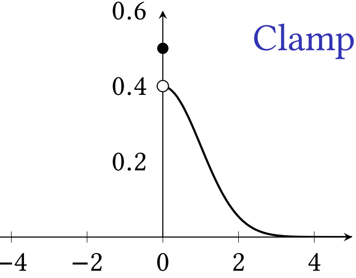
Clamped normal distribution



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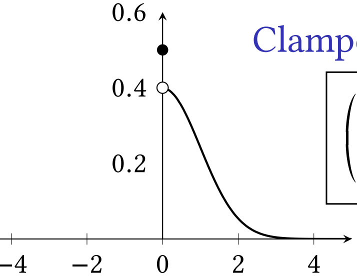

$$\begin{pmatrix}\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \quad\textbf{observe}\ (0 \le x); \\ \quad\textbf{return}\ x\}\end{pmatrix} \oplus \begin{pmatrix}\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \quad\textbf{observe}\ (x < 0); \\ \quad\textbf{return}\ 0\}\end{pmatrix}$$

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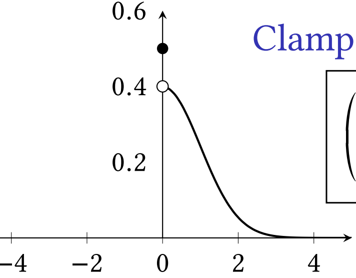

$[\![\textbf{clamped}]\!]$

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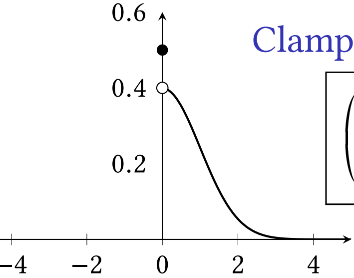
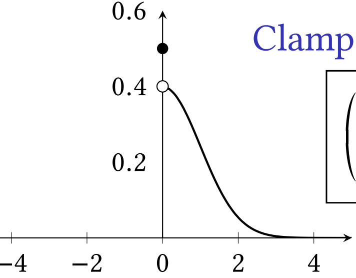


Clamped normal distribution

$$\left(\begin{array}{l}\textbf{do } \{x \leftsquigarrow \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (0 \le x); \\ \quad \textbf{return } x\}\end{array}\right) \oplus \left(\begin{array}{l}\textbf{do } \{x \leftsquigarrow \textbf{normal } 0\ 1; \\ \quad \textbf{observe } (x < 0); \\ \quad \textbf{return } 0\}\end{array}\right)$$

$$[\![\textbf{clamped}]\!] = \lambda f.\ \int_0^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot f\ x \cdot dx \ + \ 0.5 \cdot f\ 0$$

Clamped normal distribution

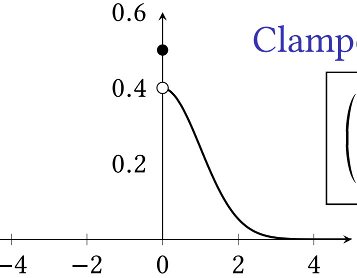


$$\begin{pmatrix}\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \quad \textbf{observe}\ (0 \le x); \\ \quad \textbf{return}\ x\}\end{pmatrix} \oplus \begin{pmatrix}\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \quad \textbf{observe}\ (x < 0); \\ \quad \textbf{return}\ 0\}\end{pmatrix}$$

$$[\![\textbf{clamped}]\!] = \lambda f.\ \int_0^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot f\ x \cdot dx \ + \ 0.5 \cdot f\ 0$$

$$= \left[\!\!\left[\begin{array}{l}\textbf{lebesgue} \otimes (\lambda t.\ \textbf{if}\ (0 \le t)\ \textbf{then}\ \textbf{exp}\ (\ldots)\ \textbf{else}\ 0) \\ \oplus \\ (\textbf{return}\ 0) \otimes \lambda t.\ \textbf{if}\ (t = 0)\ \textbf{then}\ 0.5\ \textbf{else}\ 0\end{array}\right]\!\!\right]$$

Clamped normal distribution



$$\left(\begin{array}{l}\textbf{do}\ \{x \leftarrow \textbf{normal}\ 0\ 1;\\ \quad\quad \textbf{observe}\ (0 \le x);\\ \quad\quad \textbf{return}\ x\}\end{array}\right) \oplus \left(\begin{array}{l}\textbf{do}\ \{x \leftarrow \textbf{normal}\ 0\ 1;\\ \quad\quad \textbf{observe}\ (x < 0);\\ \quad\quad \textbf{return}\ 0\}\end{array}\right)$$

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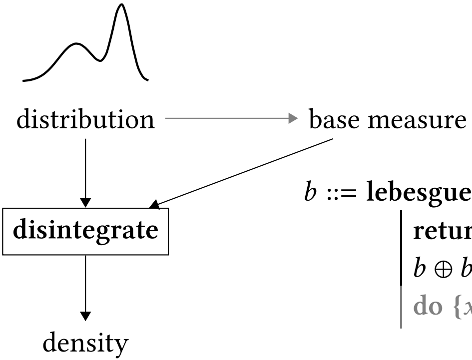
Agenda



$$b ::= \textbf{lebesgue}$$
$$\mid\ \textbf{return}\ e$$
$$\mid\ b \oplus b$$
$$\mid\ \textbf{do}\ \{x \leftsquigarrow b;\ y \leftsquigarrow b;\ \textbf{return}\ (x\,,\,y)\}$$

Multiple dimensions

$$\left[\!\left[\begin{array}{l}\textbf{do}\ \{x \leftarrow\!\sim \textbf{normal}\ 0\ 1;\\ \quad\ \ y \leftarrow\!\sim \textbf{normal}\ x\ 1;\\ \quad\ \ \textbf{return}\ (y, x)\}\end{array}\right]\!\right]$$

Multiple dimensions

$$\left[\!\!\left[\; \begin{array}{l} \textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \qquad y \leftsquigarrow \textbf{normal}\ x\ 1; \\ \qquad \textbf{return}\ (y\,,x)\} \end{array} \;\right]\!\!\right]$$

$$= \lambda f.\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}} f\ (y, x) \cdot dy \cdot dx$$

Multiple dimensions

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Sometimes move, sometimes stay

$$\left[\!\!\left[\; \begin{array}{l} \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 0\ 1; \\ \qquad y \leftsquigarrow (\mathbf{normal}\ x\ 1) \oplus (\mathbf{return}\ x); \\ \qquad \mathbf{return}\ (y\,,x)\} \end{array} \;\right]\!\!\right]$$

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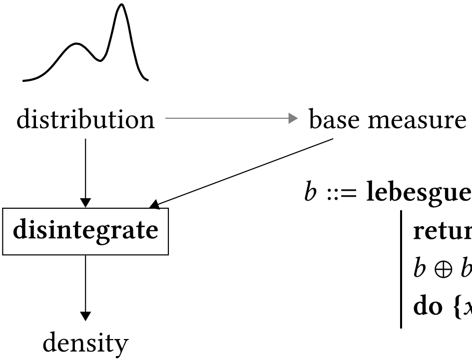
Agenda



$$
\begin{aligned}
b ::= {} & \textbf{lebesgue} \\
\mid {} & \textbf{return}\ e \\
\mid {} & b \oplus b \\
\mid {} & \textbf{do}\ \{x \leftsquigarrow b;\ y \leftsquigarrow b;\ \textbf{return}\ (x\,,\,y)\}
\end{aligned}
$$

Base measures can get complicated

Single-site proposal

```
do {p ⇜ do {x ⇜ lebesgue; y ⇜ lebesgue;
            return (x , y)};
    a ⇜ lebesgue ⊕ return (fst p);
    b ⇜ lebesgue ⊕ return (snd p);
    return (p , (a , b))}
```

Reversible-jump proposal

```
do {m ⇜ return (do {l ⇜ lebesgue; return (inl l)}
                ⊕ do {x ⇜ lebesgue;
                      y ⇜ lebesgue;
                      return (inr (x , y))});
    e ⇜ m; e′ ⇜ m;
    return (e , e′)}
```

Let’s infer bases automatically

Automatically inferring bases

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- When given a program, produce a set of density constraints

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Automatically inferring bases

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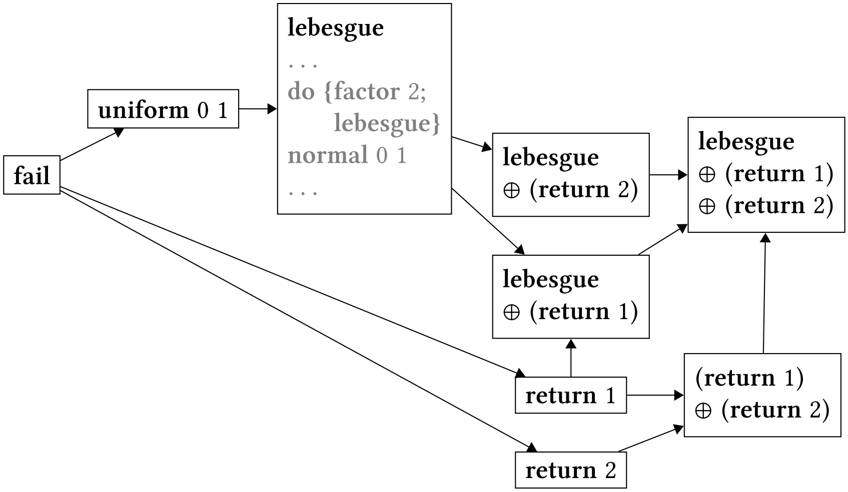
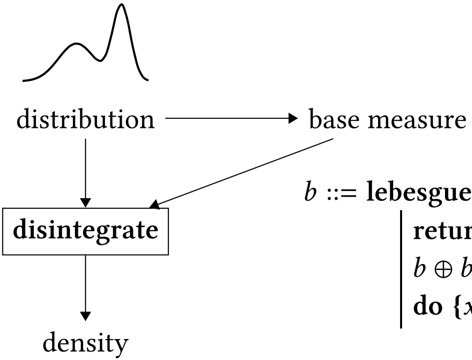
- Try to solve these constraints to produce a principal base measure

Density is a partial order



Thank you



$$
\begin{aligned}
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\end{aligned}
$$