

**Evolution of optimal Lévy-flight strategies in human mental searches**Filippo Radicchi<sup>1,\*</sup> and Andrea Baronchelli<sup>2,†</sup><sup>1</sup>*Departament d'Enginyeria Química, Universitat Rovira i Virgili, Av. Paisos Catalans 26, 43007 Tarragona, Catalunya, Spain*<sup>2</sup>*Laboratory for the Modeling of Biological and Socio-technical Systems, Northeastern University, Boston, Massachusetts 02115, USA*

(Received 11 April 2012; published 20 June 2012)

Recent analysis of empirical data [Radicchi, Baronchelli, and Amaral, *PLoS ONE* **7**, e029910 (2012)] showed that humans adopt Lévy-flight strategies when exploring the bid space in online auctions. A game theoretical model proved that the observed Lévy exponents are nearly optimal, being close to the exponent value that guarantees the maximal economical return to players. Here, we rationalize these findings by adopting an evolutionary perspective. We show that a simple evolutionary process is able to account for the empirical measurements with the only assumption that the reproductive fitness of the players is proportional to their search ability. Contrary to previous modeling, our approach describes the emergence of the observed exponent without resorting to any strong assumptions on the initial searching strategies. Our results generalize earlier research, and open novel questions in cognitive, behavioral, and evolutionary sciences.

DOI: [10.1103/PhysRevE.85.061121](https://doi.org/10.1103/PhysRevE.85.061121)

PACS number(s): 05.40.Fb, 02.50.Le, 87.23.Ge

**I. INTRODUCTION**

Lévy flights are a special class of random walk whose step lengths follow a power-law tailed distribution [1]. They have been proved to be the most efficient type of space exploration that can be adopted by a random searcher looking for scarce resources in an unknown environment [2]. Probably for this reason there is plenty of empirical evidence that movement patterns are compatible with Lévy flights in many different contexts where efficiency matters [3–12]. In particular, Lévy flights appear recurrently in the description of the motion of animals in real space (see Ref. [13] for a review). Animals explore the environment mainly for searching food resources, and it is therefore plausible to ascribe the optimality of their search strategies to a selective evolutionary process.

Recently, Radicchi *et al.* have provided empirical evidence that also human players participating in online auctions explore the bid space performing Lévy flights [14]. The exploration of the bid space represents a search process, but a purely mental one because it is performed in an abstract space. Interestingly, players adopt nearly optimal Lévy-flight exponents, in the sense that the values of the exponent used in real auctions are close to the one that maximizes their economic return. In Ref. [14], the search process in the bid space is studied as a game theoretical model, where the optimal exponent value corresponds to a Nash equilibrium [15].

Here, we propose an evolutionary model in which the reproductive fitness of the individuals is proportional to their ability to win the auctions, and we show that the values of the Lévy-flight exponents to which the model converges are very close to those measured in real data. This approach relaxes some of the assumptions made in the traditional game theoretical analysis, and deepens the understanding of the results: The optimality of the strategies adopted by bidders in online auctions can be seen as the outcome of a (evolutionary) learning process.

The paper is organized as follows. In Sec. II, we provide a detailed description of the type of auctions studied and modeled in this paper. Section III is devoted to the description of the model and its analytical treatment. In particular, Sec. III A describes the case in which all players can choose only a bid value, while, in Sec. III B, we generalize the model to the case in which players can place an arbitrary number of bids. Section III C is dedicated to the evolutionary game theoretical implementation of the model. Sections IV and V are, respectively, devoted to the description of the numerical simulations of the model and to the estimation of the computational complexity needed to simulate or solve the model. In Sec. VI, we provide a detailed description of the results of the model. Finally, in Sec. VII, we draw our final comments and considerations.

**II. LOWEST UNIQUE BID AUCTIONS**

Lowest unique bid (LUB) auctions are a recent generation of online games where the winners of the auctions may purchase expensive goods for strikingly low prices: cars, boats, and even houses can be bought for only tens or hundreds of dollars (or euros, pounds, etc.). The mechanism of the game is very simple. At the beginning of the auction, an item, whose typical market price is higher than a thousand dollars, is put up for auction. The game duration is *a priori* fixed and generally two or three weeks long. A bid can be any amount (in cents) from one cent to a certain maximum value  $M$ , generally lower than one hundred dollars. For placing a bid, a player has to pay a fee, which ranges from one to ten dollars depending on the game. During the auction, players know only the status of their own bids, meaning whether they are winning or not. None of the players knows where the others have placed their bids until the end of the game. Multiple bids on the same value from the same player are allowed, but do not influence the outcome of the auction since a bid is considered as unique when a unique player has bid that value even if more than once. When the time dedicated to the auction expires, the winner is the player who made the LUB and can finally purchase the item for the value of the winning bid. For example, if at the end of the auction bid value  $i = 1$  is occupied by two bids ( $n_1 = 2$ ), while  $n_2 = 3$ ,

\*f.radicchi@gmail.com

†a.baronchelli.work@gmail.com

$n_3 = 1$ ,  $n_4 = 2$ , and  $n_5 = 1$  are the number of bids placed on values  $i = 2, 3, 4$ , and  $5$ , respectively, then the winner is the player who has bid on value  $i = 3$  because this is the lowest bid among all unique bid values.

A similar mechanism is also used in highest unique bid (HUB) auctions. In this type of the games, the rules are the same as those of LUB auctions, with the only difference that the winning value is the unmatched bid closest to the maximal bid value admitted in the game (i.e.,  $M$ ).

LUB and HUB auctions represent competitive arenas where players perform searches for a single target whose position is determined by the bids of the whole population. It is important to stress that, during the game, players are not aware of the values on which the other players have placed their bids, and therefore the exploration of the bid space of each player can be considered independent. Also, since the cost of each bid is much larger than the natural unit of the game (one cent), the number of steps that can be performed by a single player is limited and allows only a partial exploration of the bid space. Players need therefore good strategies in order to maximize their winning chances and simultaneously maintain a limit on their investments.

In our previous work [14], we studied in detail the dynamical features of the bid space exploration performed by players in real LUB and HUB auctions. We found that the exploration of the bid space occurs in bursts: consecutive bid values are generally close to each other, but from time to time players perform longer jumps. In particular, the probability density function (PDF)  $P(d)$  of the bid change amount  $d$  ( $d$  is defined as the absolute value of the difference between two consecutive bid values) is consistent with a power-law  $P(d) \sim d^{-\alpha}$  (see Fig. 1). The exploration of the bid space is therefore consistent with a discrete version of a Lévy

flight [16]. More importantly, we found that the PDF  $g(\alpha)$  of the Lévy-flight exponents, adopted in real auctions, peaks around the average value  $\langle \alpha \rangle \simeq 1.4$  with a standard deviation of  $\sigma \simeq 0.2$ .

The empirical observation of Lévy flights provided in Ref. [14] is by far the most significant evidence of this phenomenology in natural search processes. Different from previous studies regarding biological [2,4-6,10,11] and mobility [7,8] systems where “two orders of magnitude of scaling can represent a luxury” [13], the power-law decay can be clearly observed even over four orders of magnitude. The reason is that the space is not strictly physical and movements of tens of thousands of cents can be performed at the same price as those one cent long: players explore the bid space in an effectively *superdiffusive* fashion, and steps are made at a virtually *infinite* velocity.

### III. MODELING BIDDING STRATEGIES

In this section, we provide an analytical model for LUB auctions, but the model can be easily extended also to HUB auctions. Supported by empirical evidence, in our model we assume that players explore the bid space performing Lévy flights. In our previous work [14], we provided a model based on stronger assumptions. In particular, we studied the winning chances of a player participating in an auction against a population of players using exactly the same exponent value. Here differently, we do not make any assumption regarding the choice of the exponents. We let the exponents be random variates extracted from a generic PDF, and study an evolutionary theoretical version of the game. Independent of the initial PDF, the PDF of the adopted exponents naturally evolves to a stable distribution centered around a value remarkably close to that of the exponents measured in real auctions.

#### A. First bid

We consider a population of  $N$  players whose strategies  $\alpha$  are randomly extracted from a PDF  $g(\alpha)$ . Without loss of generality, we assume that  $g(\alpha) \geq 0$  if  $\alpha \in (\alpha_1, \alpha_2)$ , and  $g(\alpha) = 0$  otherwise. During the game, each player can place bids only on integer values  $i \in [1, M]$ . Since, at the beginning of the game, none of the players knows on which values the others will bid, it is natural to think that a generic player, with strategy  $\alpha$ , sits at the leftmost site of the lattice. From this initial positions, the player places a bid on value  $i$  with probability

$$s(i, \alpha) = \frac{i^{-\alpha}}{m(\alpha)} g(\alpha), \tag{1}$$

where  $m(\alpha) = \sum_{i=1}^M i^{-\alpha}$  is the proper normalization constant. The probability that a generic player bids on value  $i$  can be calculated by simply integrating Eq. (1) as

$$p(i) = \int_{\alpha_1}^{\alpha_2} d\alpha s(i, \alpha). \tag{2}$$

After all players have bid, there will be  $n_k$  bids on the  $k$ th bid value. The probability to observe a particular configuration  $\{n\} = (n_1, n_2, \dots, n_k, \dots, n_M)$  is simply given by the

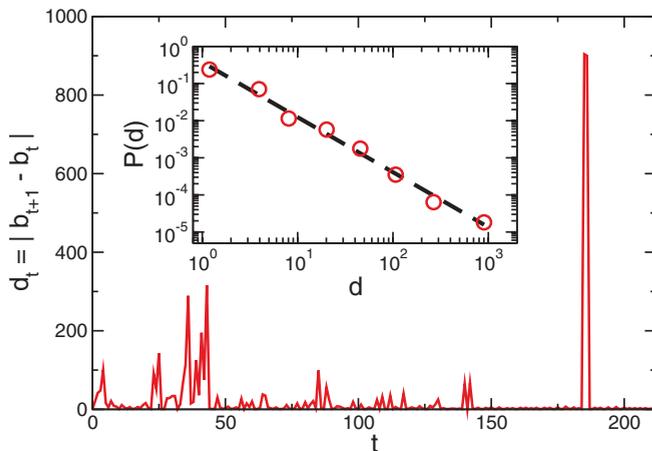


FIG. 1. (Color online) In the main panel, we plot  $d_t$  the absolute value of the difference between the two consecutive bid values  $b_{t+1}$  and  $b_t$ .  $t$  indicates the rank position of each bid in a ranking in which bids have been sorted according to their time stamp. In the inset, we plot the PDF  $P(d)$  of the bid change amounts  $d$  (red circles). The PDF is fitted with a power-law function and the best estimate of the decay exponent is  $1.5 \pm 0.1$  (black dashed line). This figure refers to the exploration of the bid space performed by a single player in a single auction. A complete analysis of the movement patterns of hundreds of players was performed in our previous work [14].

multinomial distribution

$$P(\{n\}) = N! \prod_{k=1}^M \frac{[p(k)]^{n_k}}{n_k!}, \quad (3)$$

whose weights are given by Eq. (2), and obeying the constraint  $N = \sum_{k=1}^M n_k$ . In particular, the probability that only one bid has been made on value  $i$  (i.e., the bid on value  $i$  is unique) is

$$u(i) = \sum_{\sum_{k \neq i} n_k = N-1} P(\{n\}) = N p(i) [1 - p(i)]^{N-1}. \quad (4)$$

The probability that the bid on value  $i$  is the unique and lowest bid can be exactly calculated by summing the probability of Eq. (3) over all configurations  $\{n\}$  which satisfy this constraint (i.e.,  $n_i = 1$  and  $n_j \neq 1$  for all  $j < i$ ). Unfortunately, such enumeration cannot be easily computed. A good approximation, valid for sufficiently low values of  $p(i)$  and  $u(i)$ , is to consider the uniqueness of the  $i$ th bid value as independent of the uniqueness of the other bid values and write

$$l(i) = \begin{cases} u(i), & \text{if } i = 1 \\ u(i) \prod_{k < i} [1 - u(k)], & \text{otherwise,} \end{cases} \quad (5)$$

as the probability that the bid on value  $i$  is the lowest bid among all the unique bids. The right-hand side of Eq. (5) is the product of two terms:  $u(i)$  is the probability that only one bid has been made on bid value  $i$ ;  $\prod_{k < i} [1 - u(k)]$  is the probability that none of the bid values smaller than  $i$  are occupied by a single bid (if  $i = 1$  this probability is automatically equal to one since  $i = 1$  is the minimal bid amount allowed). Finally, the probability  $w(\alpha)$  that the Lévy-flight exponent  $\alpha$  is the winning strategy in the game can be estimated, using Bayesian inference [17], as

$$w(\alpha) = \sum_{i=1}^M v(\alpha|i) l(i) = \sum_{i=1}^M \frac{s(i,\alpha) l(i)}{p(i)}. \quad (6)$$

$v(\alpha|i) = s(i,\alpha)/p(i)$  is the conditional probability to observe  $\alpha$  given  $i$ . This quantity is then convoluted over all bid values  $i$ , where the weight of each bid value is given by the probability that  $i$  is the winning bid value, that is,  $l(i)$ . Notice that Eqs. (4)–(6) explicitly depend on the number of players  $N$  and the upper bid value  $M$ . We have suppressed both variables in the notation only for clarity and shortness.

## B. Multiple bids

The same theoretical approach can be applied to the determination of the best strategy in a game where every player may perform multiple bids. We consider the simplest case in which each player bids  $T$  times, but the theory may be simply extended also to the case in which the number of bids of each player is extracted from an arbitrary PDF.

To solve this game, we need to calculate  $s_T(i)$  which stands for the probability that, in  $T$  bids, a generic player has placed a bid on value  $i$ . If  $\alpha$  is the player's strategy, the player will place the first bid on value  $i$  with probability  $q_1(i|\alpha) = i^{-\alpha}/m(\alpha)$ . For the subsequent bids, we need to define a transition matrix  $Q_\alpha$ . The generic element  $(Q_\alpha)_{ji}$  represents the probability to place a bid on value  $i$  when the previous bid was placed on

value  $j$ . In our model, we have

$$(Q_\alpha)_{ji} = \frac{|i - j|^{-\alpha}}{m_j(\alpha)}, \quad (7)$$

for all  $i$  and  $j$  in the interval  $[1, M]$ . The normalization constant  $m_j(\alpha) = \sum_{i=1}^M |i - j|^{-\alpha}$  ensures the proper definition of the transition matrix. The matrix  $Q_\alpha$  describes a random walker which follows uncorrelated Lévy flights with exponent  $\alpha$ . At a generic step  $t$ , the probability that the player with strategy  $\alpha$  sits on value  $i$  is

$$q_t(i|\alpha) = \sum_{j=1}^M (Q_\alpha)_{ji} q_{t-1}(j|\alpha). \quad (8)$$

The probability that this player placed a bid, in  $T$  bids, on value  $i$  is then

$$s_T(i|\alpha) = 1 - \prod_{t=1}^T [1 - q_t(i|\alpha)]. \quad (9)$$

The term  $1 - q_t(i|\alpha)$  counts the probability that the player has not placed a bid on value  $i$  at stage  $t$ . The probability that the player has not bid on value  $i$  at any stage is therefore the product of this single step probabilities. Finally, the probability that the player has placed a bid on value  $i$  at least once is calculated as the probability to have bid on value  $i$  an arbitrary number of times minus the probability to have never bid on value  $i$ . Notice that the model assumes that players have no memory because they are allowed to bid on the same value more than once. This is, however, unlikely to happen in the Lévy and ballistic regimes (i.e.,  $\alpha < 3$ ) [18]. Also, as in the case of real auctions, if the same player bids more than once on the same value this fact does not invalidate the uniqueness of the bid which is still considered as unique unless another player places a bid on that value.

The probability that a generic player, performing  $T$  total bids, has placed a bid on value  $i$  is then

$$p_T(i) = \int_{\alpha_1}^{\alpha_2} d\alpha s_T(i|\alpha) g(\alpha) \quad (10)$$

and can be used in place of the one appearing in Eq. (2) in order to calculate the remaining quantities  $u_T(i)$ ,  $l_T(i)$ , and  $w_T(\alpha)$  by using Eqs. (4)–(6), respectively.

## C. Evolutionary model

To understand how an optimal strategy can become shared across individuals, it is natural to adopt an evolutionary framework [19]. In this respect, our model can be implemented in terms of competing individuals that are selected on the basis of their success in the searching process. In the spirit of the fundamental Moran process [20], at the end of each game, the winner of the auction generates an offspring to which her/his search exponent  $\alpha$  is transmitted. The new individual enters the population endowed with an exponent  $\alpha + \xi$  (with  $\xi$  random mutation), while a randomly extracted individual is removed in order to maintain the population size constant. Basically, the PDF of the winning exponents of the former generation corresponds to the fitness function of the evolutionary model.

### 1. Absence of mutations

Let us first consider the case in which losers copy the strategy of the winners without errors. Imagine to have  $N$  players at each generation. They play the game by performing  $T$  bids each. Denote with  $e$  the number of the generation. At the beginning, we set  $e = 1$ . Then we follow the scheme:

- (1) Players randomly pick strategies  $\alpha$  from the PDF  $g^{(e)}(\alpha)$ .
- (2) They play the game. The result is the PDF  $w_T^{(e)}(\alpha)$ , which quantifies the probability that  $\alpha$  was a winning strategy.
- (3) Set  $g^{(e+1)}(\alpha) = w_T^{(e)}(\alpha)$ , increment  $e \rightarrow e + 1$ , and go back to point 1.

This procedure describes the evolution of a population of players under repeated games. Setting  $g^{(e+1)}(\alpha) = w_T^{(e)}(\alpha)$  ensures that the players of the new generation have the tendency to pick winning strategies instead of losing ones. This can be better understood by writing the master equation

$$g^{(e+1)}(\alpha) - g^{(e)}(\alpha) = w_T^{(e)}(\alpha)[1 - g^{(e)}(\alpha)] - [1 - w_T^{(e)}(\alpha)]g^{(e)}(\alpha), \quad (11)$$

from which one can easily obtain  $g^{(e+1)}(\alpha) = w_T^{(e)}(\alpha)$ . Equation (11) tells us that the variation in the population of players with exponent  $\alpha$  increases as the probability that a player with strategy  $\alpha$  wins [ $w_T^{(e)}(\alpha)$ ] times the probability to have other players with strategies different from  $\alpha$  [ $1 - g^{(e)}(\alpha)$ ], and decreases as the probability that a player with strategy different from  $\alpha$  wins [ $1 - w_T^{(e)}(\alpha)$ ] times the probability to find a player with strategy equal to  $\alpha$  [ $g^{(e)}(\alpha)$ ]. In other words, the probability of reproduction of a player is proportional to the ability of the player to win the game. The evolution rules resemble a Moran process where selection is made according to a fitness function here defined as the probability to win the game [20].

### 2. Random mutations

A more natural assumption is to formulate a model where, each time a losing player changes strategy, he copies the exponent of the winner plus some random variation. Assume that the variation  $\xi$  is randomly extracted from a PDF  $y(\xi, \alpha, \vec{\mu})$ , explicitly dependent on  $\alpha$  and a set of parameters  $\vec{\mu}$ . The master equation describing the evolution becomes

$$g^{(e+1)}(\alpha) - g^{(e)}(\alpha) = + \int d\beta \int d\xi \delta(\alpha - \beta + \xi) y(\xi, \beta, \vec{\mu}) \times w_T^{(e)}(\beta)[1 - g^{(e)}(\beta)] - [1 - w_T^{(e)}(\alpha)]g^{(e)}(\alpha),$$

with  $\delta(x) = 1$  if and only if  $x = 0$ , while  $\delta(x) = 0$  otherwise. The gain term stands for the probability that the generic exponent  $\beta$  represents the winning strategy [ $w_T^{(e)}(\beta)$ ] times the probability to have other players with exponents different from  $\beta$  [ $1 - g^{(e)}(\beta)$ ]. This term is then convoluted with the noise over all possible values of  $\beta$ ; this quantifies the probability that a random mutation changes the exponent from  $\beta$  to  $\alpha$  [ $\delta(\alpha - \beta + \xi)$ ]. The loss term is simply the probability that  $\alpha$  is not the winning strategy [ $1 - w_T^{(e)}(\alpha)$ ] and therefore players with exponent  $\alpha$  move away from it at rate  $g^{(e)}(\alpha)$ . The  $\delta$  function imposes the condition  $\alpha - \beta + \xi = 0$  and the

preceding equation reduces to

$$g^{(e+1)}(\alpha) = w_T^{(e)}(\alpha) g^{(e)}(\alpha) + \int d\beta y(\beta - \alpha, \beta, \vec{\mu}) \times w_T^{(e)}(\beta)[1 - g^{(e)}(\beta)]. \quad (12)$$

The PDFs of the adopted strategies converge to stability whenever an  $e$  exists for which  $g^{(e+1)}(\alpha) = g^{(e)}(\alpha)$ , which automatically implies that  $w_T^{(e+1)}(\alpha) = w_T^{(e)}(\alpha)$ .

## IV. NUMERICAL SIMULATIONS

The former analytical formulation of the model does not allow us to obtain explicit expressions regarding the distributions of the winning strategies. The various equations can, in fact, be only numerically integrated to provide a solution of the model. Moreover, some of the equations contain approximations, and it is therefore worth asking whether the solutions obtained with the numerical integration of the equations are compatible with those obtained by directly simulating the model.

Simulating our model is straightforward. In each simulation, we use the following scheme:

- (1) Extract the exponent values of each of the  $N$  players from the given PDF  $g(\alpha)$ .
- (2) Simulate the game: for each of the  $N$  players, extract  $T$  integer bid values from the corresponding power-law distribution, and determine the winner of the auction (the player who made the LUB) on the basis of these extractions.

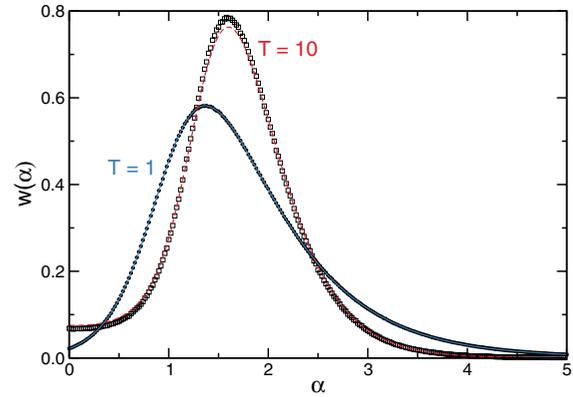


FIG. 2. (Color online) Probability density function of the winning strategy  $w(\alpha)$  in the case in which players' strategies are randomly extracted from a uniform PDF  $g(\alpha)$  defined over the interval  $(0, 5)$ . We set  $N = 100$  and  $M = 1000$  (parameter values similar to those observed in real auctions [14]), and consider for simplicity only the case of a single bid  $T = 1$  and the case in which all players perform  $T = 10$  bids. We show in both cases the comparison between the results obtained with numerical simulations ( $T = 1$ , filled circles;  $T = 10$ , empty squares) and those obtained with the numerical integrations of the various equations [ $T = 1$  full (lower) line,  $T = 10$  dashed (upper) line]. Note that the optimal value of  $\alpha$  depends on  $T$  and also on the parameters values  $N$  and  $M$ . However, this is a weak dependence and the peak value of the probability density function of the winning strategy is in the range 1.2–1.7 for a wide range of possible values of  $N$  and  $M$  and a moderately broad range of values of  $T$ .

In the case of the evolutionary game model, at the end of the game we need to change the exponent value of one of the losing players, by copying (with or without random mutations) the strategy of the winner. We then repeat the game. A generation corresponds to  $N$  exponent changes. The PDFs of the winning strategies of each generation are computed by repeating the entire procedure many times.

## V. COMPUTATIONAL COMPLEXITY

One could ask, Why use a complicated and approximated set of equations instead of simple and straightforward numerical simulations? The reason is that the computational time required for the numerical integration of the model's equations is much much lower than the one needed for obtaining good

estimates with numerical simulations. For clarity, we provide here an estimation of the computational complexity required in both approaches to the solution of the model.

Consider first the case  $T = 1$  (i.e., players make a single bid). In the case of numerical simulations with fixed values of the exponents, the time required to simulate the game grows as  $M + N \log(M)G$ .  $M$  is the number of possible bid values and indicates also the computational time required to calculate the transition matrix from the starting position in the bid space (i.e., the origin of the lattice) to all  $M$  possible bid values.  $N$  is the number of players. The bid value on which each player places a bid can be calculated in a time that scales as  $\log(M)$ . Finally  $G$  is the number of times that we need to simulate the same auction model in order to obtain a good estimation of the PDF  $w(\alpha)$ . The computational complexity of the

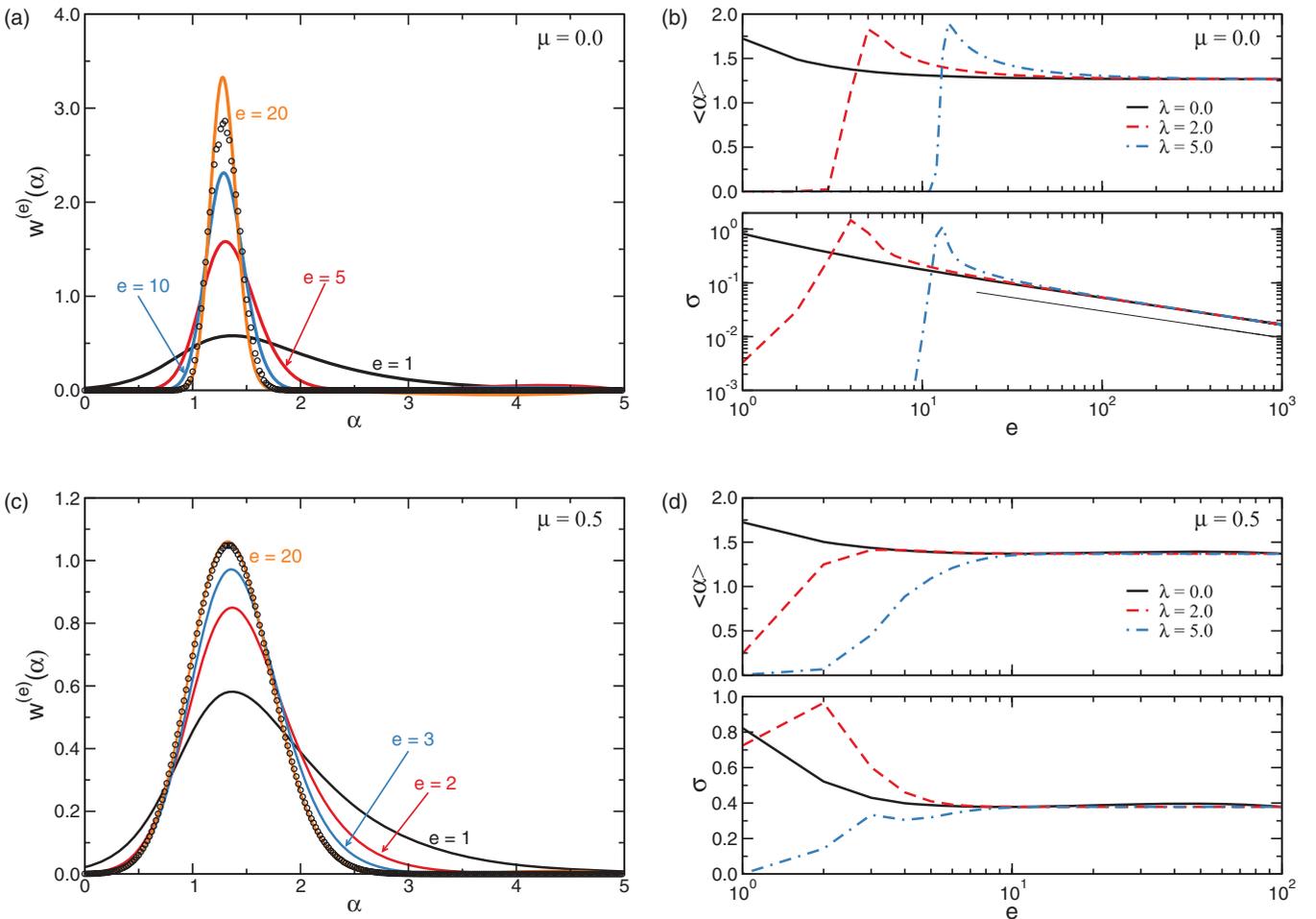


FIG. 3. (Color online) Probability distribution function  $w^{(e)}(\alpha)$  of the winning strategies at generation  $e$ . Here we set  $M = 1000$ ,  $N = 100$ , and  $T = 1$ . The mutation  $\xi$  is randomly extracted from a uniform distribution centered in zero and width  $2\mu$ : we consider the cases  $\mu = 0.0$  (panels a and b, absence of mutation) and  $\mu = 0.5$  (panels c and d). In a and c, the starting distribution for the exponents is  $g^{(1)}(\alpha) = 1/5$  if  $\alpha \in (0,5)$  and zero otherwise. Already after one generation ( $e = 1$ , lower black line), the peak of the distribution is around  $\alpha \simeq 1.5$ . As the number of generations increases, the distribution becomes more and more peaked around a specific value of  $\alpha$ , and reaches a stationary distribution. The asymptotic distribution has finite width for  $\mu > 0$ , while it is a  $\delta$  function for  $\mu = 0$ . For comparison, we show also the results obtained numerically after  $e = 20$  generations (black circles). In b and d, we consider initial distributions for the exponents of the type  $g^{(1)}(\alpha) \sim \alpha^{-\lambda}$  if  $\alpha \in (0,5)$  and  $g^{(1)}(\alpha) = 0$  otherwise. We show the results for the three cases:  $\lambda = 0$  (uniform),  $\lambda = 2$  (power law), and  $\lambda = 5$  (exponential). The asymptotic distribution does not depend on the initial distribution, and the peak value of the asymptotic PDF does not depend on  $\mu$ . Independently of the value of  $\lambda$ , as the number of generations  $e$  increases, the average value  $\langle \alpha \rangle$  and the standard deviation  $\sigma$  approach the same stationary values:  $\langle \alpha \rangle \simeq 1.3$  for any  $\mu$ , while  $\sigma = 0$  (as  $1/\sqrt{e}$ , thin black line in panel b) for  $\mu = 0.0$  and  $\sigma \simeq 0.4$  for  $\mu = 0.5$ .

numerical solutions of equations is  $M$ , since the computation of Eqs. (1), (2), and (4)–(6) require a computational time that grows as  $M$ .

For general values of  $T$ , the computational complexity of numerical simulations is simply incremented by a factor  $T$  and grows therefore as  $TM + TN \log(M)G$ . The time required for the numerical integration of the equation differently grows as  $TM^2$ . The most computationally expensive calculation is the one of Eq. (8), which requires a time that grows as  $M^2$ , and this computation has to be repeated  $T$  times.

## VI. RESULTS

In all the results, we consider the values  $N = 100$  and  $M = 1000$ . The choice of these parameter values is justified because they are of the same order of magnitude as those measured in real auctions [14]. In Fig. 2, we plot the PDF  $w(\alpha)$  for  $T = 1$  and  $T = 10$ . In both cases,  $g(\alpha) = 1/5$  if  $\alpha \in (0, 5)$ , while  $g(\alpha) = 0$  otherwise. Players randomly choose strategies that correspond to ballistic motion ( $\alpha \leq 1$ ), diffusive motion ( $\alpha > 3$ ) and superdiffusive motion or Lévy flight ( $1 < \alpha \leq 3$ ). Assigning a flat initial distribution corresponds to assuming

that players do not know which strategy is better for winning the game, and all strategies are therefore *a priori* equivalent. It is interesting to see that already in this situation there is a clear advantage for players that perform Lévy flights and whose exponents are in the range 1.2–1.5. It is also worth noting that the solutions obtained with the numerical integration of the equations are perfectly consistent with the results obtained with numerical simulations.

More interesting is the case of evolutionary games. For simplicity, here we consider the case in which the distribution of the random mutations is a rectangular window of width  $2\mu$ : if  $\alpha$  is the winning strategy, the new exponent is equal to a random number taken from the uniform distribution into the interval  $(\alpha - \mu, \alpha + \mu)$ . In the case in which the possible values of the exponents are bounded in the interval  $(\alpha_1, \alpha_2)$ , we have to include the effect of the boundaries and write

$$y(\xi, \alpha, \alpha_1, \alpha_2, \mu) = \begin{cases} [l_1 + l_2]^{-1}, & \text{if } \xi \in (-l_1, l_2) \\ 0, & \text{otherwise,} \end{cases} \quad (13)$$

where  $l_1 = \min(\alpha - \alpha_1, \mu)$  and  $l_2 = \min(\alpha_2 - \alpha, \mu)$ . We show in Fig. 3 the results valid for  $T = 1$  and in Fig. 4 those obtained for  $T = 10$ . These figures show results that are

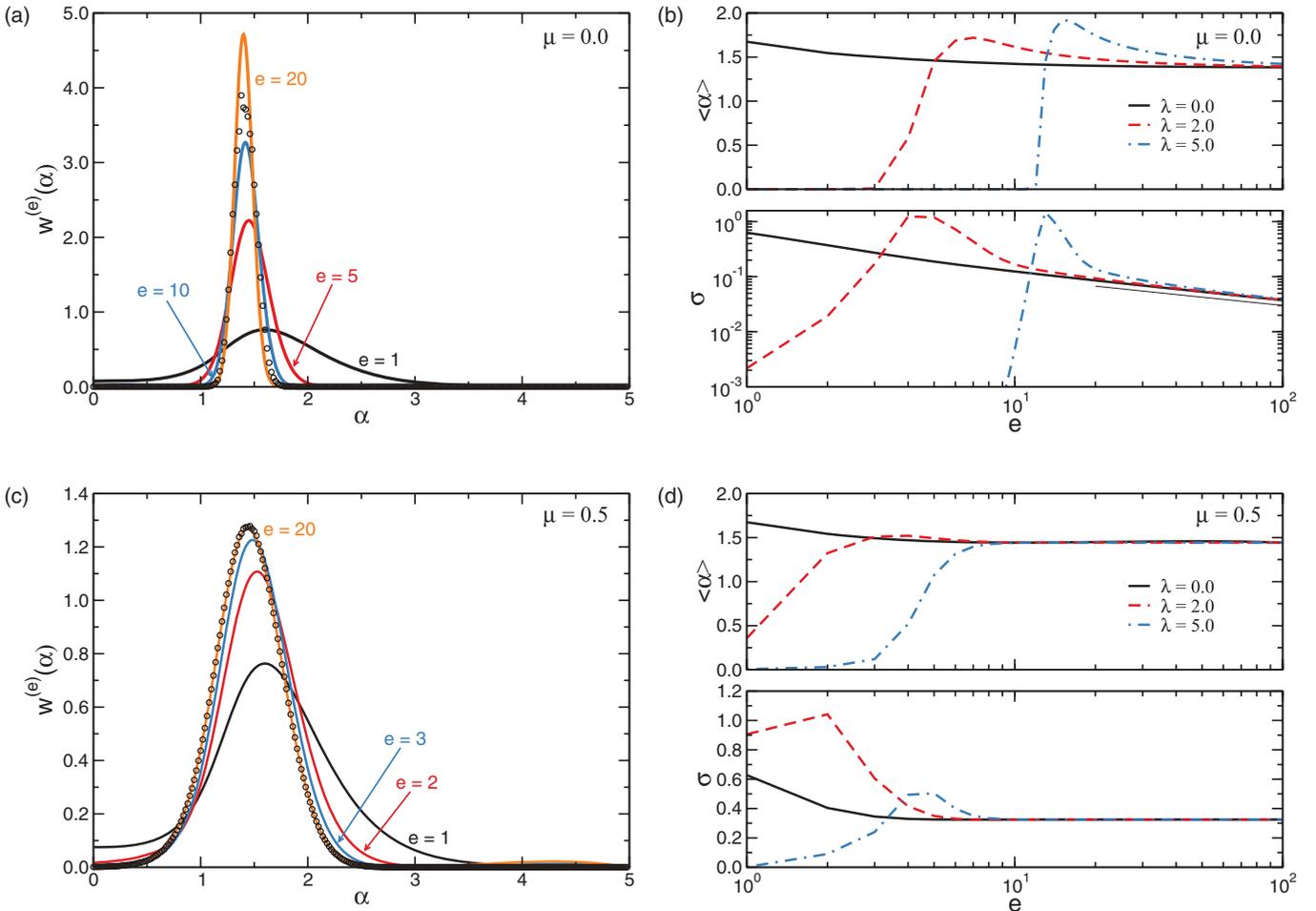


FIG. 4. (Color online) Same as Fig. 3, but with different parameter values. Here we set  $M = 1000$ ,  $N = 100$ , and  $T = 10$ , and consider  $\mu = 0.0$  in panels a and b, while  $\mu = 0.5$  in panels c and d. The asymptotic PDF of the winning strategies have average values  $\langle \alpha \rangle \simeq 1.5$  for any value of  $\mu$ . The standard deviation is  $\sigma = 0$  (as  $1/\sqrt{e}$ ) for  $\mu = 0.0$ , and  $\sigma = 0.3$  for  $\mu = 0.5$ . For any value of  $\mu$ , the best strategy (i.e., the peak of the PDF) is placed at  $\alpha^* \simeq 1.5$ .

slightly different but qualitatively identical. The asymptotic PDF  $w^{(\infty)}(\alpha)$  of the winning strategies does not depend on the initial distribution  $g^{(1)}(\alpha)$ . In the absence of mutation, the limiting PDF is a  $\delta$  function centered around an optimal strategy  $\alpha^*$ . The convergence to such a PDF is only asymptotic, since the width of the distribution goes to zero as  $1/\sqrt{e}$  and is therefore finite at any finite generation  $e$ . If mutations are allowed (i.e.,  $\mu > 0$ ),  $w^{(\infty)}(\alpha)$  is reached after a finite number of iterations. The asymptotic distribution has a finite width. The number of iterations required to reach stability depends on the initial PDF and the mutation rate  $\mu$ , while its shape only on  $\mu$ . In particular, the peak of the PDF is still at  $\alpha^*$ , the same value as the one measured in the absence of mutations. The value of  $\alpha^*$  depends on the parameters of the model  $N$ ,  $M$ , and  $T$ ; but for values consistent with those of real auctions ( $N = 50$  to  $N = 200$ ,  $M = 500$  to  $M = 10000$ , and  $T = 1$  to  $T = 100$ ),  $\alpha^*$  ranges from 1.2 to 1.7.

## VII. CONCLUSIONS

A large wealth of empirical evidence suggests that animals have a specific, and apparently innate, strategy to search an unknown physical environment [13]. The distances between two consecutive positions in the space are distributed according to a power-law probability density function. More interestingly, the exponent of the power-law distribution is close to the one that guarantees the most efficient search in an environment with scarce resources. Evolutionary considerations account for the high efficiency of the searching strategies of animals, since in a competitive environment only the fittest are able to survive and reproduce.

A similar behavior has been observed also in how human players explore the bid space in online auctions [14]. In this case, the environment is not physical but abstract. Nevertheless players adopt searching strategies for the winning bid value that are close to optimality: bid change amounts are power-law distributed, and the exponents of the power laws are close to the value that can guarantee the highest winning chances.

In this paper, we have provided a different interpretation of this empirical evidence and used evolutionary considerations to explain the optimality of the observed exponents. We have introduced a Moran-like model in which the reproductive fitness of the players is proportional to their success in searches [20]. The player winning the auction is reproduced, in the sense that the strategy of the winner is transmitted to another randomly extracted individual. We have considered the cases of error-free reproduction and of transmission with mutation. We have described the model analytically through a set of equations whose numerical solution is in excellent agreement with direct agent-based simulations. We have shown that the model is extremely robust with respect to the choice of the different parameters, producing results in good agreement with the ones observed in the empirical data.

In summary, looking at activity patterns in the web [21], our previous work [14] suggested that humans and other animals share the same, apparently innate, strategy to search in an unknown physical or mental environment. Here we have shown that an evolutionary approach allows one to account for the optimality of the observed exponents, in agreement with the view according to which the ability to understand and be effective in the natural world is likely to be innate [22]. This is the case, for example, of locomotion and perceptual-motor control [23], hunting and foraging [24], or nest building [25]. We have provided an example that concerns a mental search process. While it is well known that humans share the intuition that numbers map into space [26], our work indicates that they might have developed an innate knowledge about the best way to move in it.

## ACKNOWLEDGMENTS

The authors are indebted to Luís A. N. Amaral for helpful discussions on the subject of this article. F.R. acknowledges support from the Spanish Ministerio de Ciencia e Innovación through the Ramón y Cajal program.

- 
- [1] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, *Nature* **363**, 31 (1993).
  - [2] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, *Nature* **401**, 911 (1999).
  - [3] R. Mantegna and H. E. Stanley, *Nature* **376**, 46 (1995).
  - [4] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, *Nature* **381**, 413 (1996).
  - [5] F. Bartumeus, F. Peters, S. Pueyo, C. Marrasé, and J. Catalan, *Proc. Natl. Acad. Sci. USA* **100**, 12771 (2003).
  - [6] G. Ramos-Fernandez, J. L. Mateos, O. Miramontes, G. Cocho, H. Larralde, and B. Ayala-Orozco, *Behav. Ecol. Sociobiol.* **55**, 223 (2004).
  - [7] S. Bertrand, J. M. Burgos, F. Gerlotto, and J. Atiquipa, *J. Mar. Sci.* **62**, 477 (2005).
  - [8] D. Brockmann, L. Hufnagel, and T. Geisel, *Nature* **439**, 462 (2006).
  - [9] T. Rhodes and M. T. Turvey, *Physica A* **385**, 255 (2005).
  - [10] D. W. Sims *et al.*, *Nature* **451**, 1098 (2008).
  - [11] N. E. Humphries *et al.*, *Nature* **465**, 1066 (2010).
  - [12] G. M. Viswanathan, *Nature* **465**, 1018 (2010).
  - [13] G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz, *Phys. Life Rev.* **5**, 133 (2008).
  - [14] F. Radicchi, A. Baronchelli, and L. A. N. Amaral, *PLoS ONE* **7**, e029910 (2012).
  - [15] J. Nash, *Proc. Natl. Acad. Sci. USA* **36**, 48 (1950).
  - [16] D. H. Hughes, M. F. Shlesinger, and E. W. Montroll, *Proc. Natl. Acad. Sci. USA* **78**, 3287 (1981).
  - [17] D. J. C. MacKay, *Information Theory, Inference, and Learning* (Cambridge University, Cambridge, England, 2003).
  - [18] G. Berkolaiko and S. Havlin, *Phys. Rev. E* **55**, 1395 (1997).
  - [19] M. A. Nowak, *Evolutionary Dynamics: Exploring the Equations of Life* (Belknap, Cambridge, MA, 2006).

- [20] P. A. P. Moran, *Math. Proc. Cambridge* **54**, 60 (1958).
- [21] D. Lazer *et al.*, *Science* **323**, 721 (2009).
- [22] N. Chater and M. H. Christiansen, *Cognitive Sci.* **34**, 1131 (2010).
- [23] R. M. Alexander, *Principles of Animal Locomotion* (Princeton University, Princeton, NJ, 2003).
- [24] D. W. Stephens, J. S. Brown, and R. C. Ydenberg, *Foraging: Behavior and Ecology* (University of Chicago, Chicago, 2007).
- [25] S. D. Healy, P. Walsh, and M. Hansell, *Curr. Biol.* **18**, R271 (2008).
- [26] S. Dehaene, V. Izard, E. Spelke, and P. Pica, *Science* **320**, 1217 (2008).