# CSCI 241H:
## HOMEWORK 7
Only questions 1-4 will be graded. Show your work.

1. Let $T$ be a complete binary tree, with root node $k$. Let $l_t$ be the number of nodes in the left subtree of $k$ (the left subtree is the tree which has its left child as its root). Prove by induction that the number of nodes in $l_t$ is 1 less than the number of leaves in the tree. Do not prove or argue what each quantity is, work on the relationship between them in an inductive fashion (you can do that for the base case). You will lose points for statements such as "well, the tree has $q$ leaves because ..." or "we know that the size of the subtree is ...". I am not interested in how many leaves, or how many in the subtree – I want to be shown that one is 1 more than the other.

   We induct on $h$, the number of levels of the tree. For $h = 1$, we have one root and two children. There are 2 leaves, and the left subtree of the root has 1 child, so the claim holds. Let's assume then that it holds for trees with $k$ levels.

   Now consider a tree with $k + 1$ levels. It is composed of one root whose left subtree and right subtree are each complete binary trees of $k$ levels. Let $l$ be the number of leaves in each subtree. Then our full tree has $2l$ leaves. Also, by the inductive hypothesis, the subtree's left subtree has $l - 1$ vertices. Thus, the left subtree must have two of these plus a root, $2l - 1$ nodes. So, the number of nodes in the left subtree is still one less than the number of leaves.

2. A $d-$regular graph is an undirected graph where each vertex is of degree exactly $d$. Is it possible to have a 3-regular graph on 5 nodes? Prove. How about on 6 nodes?

   No, and yes. The sum of the degrees must be odd so the first case is not possible. The second is possible; just draw a triangular prism.

3. Prove that the sum of the indegrees of the vertices of a directed graph is equal to the sum of the outdegrees.

   Prove by induction on the number of edges. Quick version: each added edge increases both sums by one.

4. Show that in an undirected graph with at least two vertices there must be two vertices which have the same degree.

The maximum degree is $n-1$. Assume there is no node with degree 0. Then there are $n$ vertices and $n-1$ possible degrees, then two vertices must have the same degree. What happens if there is a node with degree 0? Then the maximum degree is $n-2$, and the same argument can be applied.

5. Let $G$ be a bipartite graph where every vertex has the same degree ($G$ is $d-$regular). Show that the sizes of the two partitions are equal.

   So there are $n$ vertices, $k$ in the first partition and $n-k$ in the second. The sum of the degrees in the first partition is equal to the number of edges in the graph. The same is true for the second partition. But then they are the same, and, since the degree of each node is the same, the two partitions must have the same number of nodes. $k = n - k = n/2$

6. Let $G$ be a bipartite graph with $n$ vertices and $m$ edges. Show that $e \leq v^2/4$.

   What is the maximum number of edges in $G$? Let the two partitions be sizes $k$ and $n-k$. Then, there can be at most $kn - k^2$ edges. This expression is maximized (remember calculus) when its first derivative is 0, where $2k - n = 0$ Thus $k = n/2$

7. This is a tough one, I think: A Hamiltonian cycle is cycle through a graph (a sequence of edges which will leave you at your starting point), which visits every node exactly once). Show by induction that, for $d > 1$, any hypercube of degree (dimension) $d$ has a Hamiltonian cycle.

   We will do this in class.