CSCI 241H, Midterm 1

Pick any 5 questions; each question is 20 points. If you attempt all 6, I will drop the one where you got the lowest score.

1. Mary goes to the doctor if she's feeling neither energetic nor sharp. If she has a headache and blurred vision she has a migraine attack. If she does not have a headache she feels energetic. If she does not have blurred vision she feels sharp. She is not having a migraine attack. So, is it possible that she has gone to the doctor today? Prove your answer using rules of inference. Show every step, but you don't need to write down the name of the rule you're using (but feel free to put it down if you feel like it).

   D: Mary goes to the doctor E: Mary is energetic S: Mary is sharp H: Mary has a headache B: Mary has blurred vision M: Mary has a migraine attack.

   This is very similar to the Superman problem.

   We have:

   1. $\neg E \wedge \neg S \rightarrow D$.
   2. $H \wedge B \rightarrow M$.
   3. $\neg H \rightarrow E$.
   4. $\neg B \rightarrow S$.
   5. $\neg M$.
   6. $\neg H \vee \neg B$ (2,5)
   7. $E \vee S$ (3,4,6)
   8. $(\neg E \wedge \neg S)$ is false. (7)

   Since F implies anything, 1 is satisfied regardless of the value of D. So she can be at the doctor's.

2. Consider the infinite set $S = \{1, 2, 4, 8, 16, 32, 64, \ldots\}$. Argue that the cardinality of S is the same as the cardinality of the set of positive integers.

   Define $f : Z^+ \rightarrow S$ as $f(i) = 2^{i-1}$ This will map every positive integer to a distinct element of S. Also note that all elements of S will be mapped to a positive integer (namely, x to $\log x + 1$). Thus f is a bijection.

Alternative solution: show two functions, one in each direction. f is the same as above, g is the identity function from S to the positive integers. Since both are one to one, the claim follows.

3. Argue that $\exists x P(x) \wedge \exists x Q(x)$ is not equivalent to $\exists x(P(x) \wedge Q(x))$.

   Let $P(x)$ be x is even, $Q(x)$ be x is odd. Given a domain that consists of odd and even numbers, the first statement holds but the second does not.

4. Use proof by contradiction to show that $\sqrt{3}$ is irrational.

   This is the same as $\sqrt{2}$. Assume $\sqrt{3}$ is rational, then it can be written as a/b for relatively prime a, b. Then we have $a^2 = 3b^2$ and thus a^2 is divisible by 3. Then a must be divisible by 3, and we can write $a = 3d$ for some integer d. Then, $9d^2 = 3b^2$, which means b^2 is divisible by 3. This means that b itself is divisible by 3, which contradicts a and b being relatively prime.

5. Argue that for any three sets A, B, C, if $A \cap B \subseteq C$ then $A - C \subseteq A - B$. Hint: if $x \in A - C$ then what can you say about x? What do you need to show about x? Do not give a proof by Venn diagram, even though you can use a Venn diagram for your own intuition.

   I will show inferences without the names of the rules. We have, for those x in $A - C$,

   1. $x \in A \wedge x \notin C$

   2. $x \notin C$ (1)

   3. $x \in A \cap B \rightarrow x \in C$ (given)

   4. $x \notin C \rightarrow (x \notin A \vee x \notin B)$ (3)

   5. $(x \notin A \vee x \notin B)$ (2,4)

   6 $x \in A$ (1)

   7. $x \notin B$ (5,6)

   8. $x \notin B \wedge x \in A$ (6, 7)

   9. $x \in A - B$ (8)

6. Consider the recurrence $f(i) = f(i-1) + 2f(i-2)$. If $f(0) = 1$ and $f(1) = 2$, give a closed form expression for $f(n)$. Note that a summation is not a closed form expression. Show your work.

   The recurrence is $1, 2, 4, 8, 16, \ldots$ thus $f(n) = 2^n$.