Lifted inference: normalizing loops by evaluation

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I	Disease is infectious?
$S(a)$	Alice is sick?
$S(b)$	Bob is sick?
$S(c)$	Carol is sick?

I	Disease is infectious?
$S(a)$	Alice is sick?
$S(b)$	Bob is sick?
$S(c)$	Carol is sick?

$$\frac{\Pr(S(b))}{\Pr(\neg S(b))}?$$

Probabilities represent uncertainty.

I	Disease is infectious?
$S(a)$	Alice is sick?
$S(b)$	Bob is sick?
$S(c)$	Carol is sick?

$$\frac{\Pr(S(b) \wedge \neg S(a))}{\Pr(\neg S(b) \wedge \neg S(a))} ?$$

Conditional probabilities represent prior knowledge.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t

Possible worlds as in modal logic.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

Possible worlds as in modal logic, but weighted numerically.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\Big(\prod_{\neg I \wedge \neg S(x)} 9\Big)$$

Possible worlds as in modal logic, but weighted numerically.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\Big(\prod_{\neg I \wedge \neg S(x)} 9\Big) \times \Big(\prod_{I \wedge \neg S(x)} 4\Big)$$

Possible worlds as in modal logic, but weighted numerically.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\overbrace{\Bigl(\prod_{\neg I \wedge \neg S(x)} 9\Bigr) \times \Bigl(\prod_{I \wedge \neg S(x)} 4\Bigr)}^{\text{Weight}}$$

```
let I = dist [(1, true);
              (1, false)]
in let S(x) = dist [(1, true);
                    (if I then 4 else 9, false)]
   in ...
```

Want to express generative model: random choice as side effect.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\overbrace{\Big(\prod_{\neg I \wedge \neg S(x)} 9\Big) \times \Big(\prod_{I \wedge \neg S(x)} 4\Big)}^{\text{Weight}}$$

```
create table Weights (W, Weight) as
select W, (select product(case when I then 4 else 9 end)
           from Infectious join Sick using (W)
           where W = Worlds.W and not S)
from Worlds
```

Express a wide variety of queries. Typically:

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\text{Tabulate} \sum_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\} \\ \text{for each } z\neq b}} \overbrace{\Big(\prod_{\neg I\wedge\neg S(x)} 9\Big) \times \Big(\prod_{I\wedge\neg S(x)} 4\Big)}^{\text{Weight}}$$

```
select S, sum(Weight)
from Weights
     join Sick as B using (W)
where B.Person = 'b'
group by B.S
```

Group worlds and sum weights.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\text{Tabulate}_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\}\\ \text{for each } z\neq b}} \overbrace{\Big(\prod_{\neg I\wedge\neg S(x)} 9\Big) \times \Big(\prod_{I\wedge\neg S(x)} 4\Big)}^{\text{Weight}} \times (\text{if } S(a) \text{ then } 0 \text{ else } 1)$$

```
select S, sum(Weight)
from Weights
     join Sick as B using (W) join Sick as A using (W)
where B.Person = 'b' and A.Person = 'a' and not A.S
group by B.S
```

Group worlds and sum weights, filtered by prior knowledge.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\text{Tabulate} \sum_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\} \\ \text{for each } z\neq b}} \overbrace{\Big(\prod_{\neg I\wedge\neg S(x)} 9\Big) \times \Big(\prod_{I\wedge\neg S(x)} 4\Big)}^{\text{Weight}} \times (\text{if } S(a) \text{ then } 0 \text{ else } 1)$$

Query plan

$$= \text{Tabulate} \sum_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \prod_{x} \text{let } f(S) = (\text{if } \neg I \wedge \neg S \text{ then } 9 \text{ else } 1) \cdot (\text{if } I \wedge \neg S \text{ then } 4 \text{ else } 1) \cdot (\text{if } x = a \wedge S \text{ then } 0 \text{ else } 1)$$
$$\text{in if } x = b \text{ then } f(S(b)) \text{ else } f(\mathtt{t}) + f(\mathtt{f})$$

Distributivity turns sum of 2^{n-1} products into product of n sums.

I	f	f	f	f	f	f	f	f	t	t	t	t	t	t	t	t
$S(a)$	f	f	f	f	t	t	t	t	f	f	f	f	t	t	t	t
$S(b)$	f	f	t	t	f	f	t	t	f	f	t	t	f	f	t	t
$S(c)$	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t
	729	81	81	9	81	9	9	1	64	16	16	4	16	4	4	1

$$\text{Tabulate}_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\} \\ \text{for each } z\neq b}} \overbrace{\Big(\prod_{\neg I\wedge\neg S(x)} 9\Big) \times \Big(\prod_{I\wedge\neg S(x)} 4\Big)}^{\text{Weight}} \times (\text{if } S(a) \text{ then } 0 \text{ else } 1)$$

Query plan

$$= \text{Tabulate}_{S(b)\in\{\mathtt{f},\mathtt{t}\}} \sum_{I\in\{\mathtt{f},\mathtt{t}\}} \prod_{x} \underbrace{\begin{array}{l} \text{let } f(S) = (\text{if } \neg I \wedge \neg S \text{ then } 9 \text{ else } 1) \\ \qquad \cdot (\text{if } I \wedge \neg S \text{ then } 4 \text{ else } 1) \\ \qquad \cdot (\text{if } x = a \wedge S \text{ then } 0 \text{ else } 1) \\ \text{in if } x = b \text{ then } f(S(b)) \text{ else } f(\mathtt{t}) + f(\mathtt{f}) \end{array}}_{\text{Boring loop}}$$

Lift probabilities *generally?*

Propositional ⟶ First-order

Semantic foundation — Exploit symmetry

Lift probabilities *generally?*

Propositional ⟶ First-order

Semantic foundation

Exploit symmetry

Reasoning system?

Inference algorithm?

Query plan? Yes!

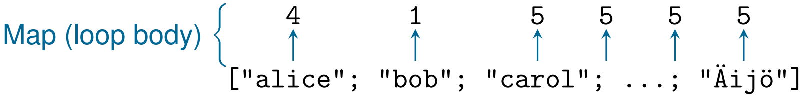
Outline

- **MapReduce loops in sublinear time**
- Normalizing loop bodies by evaluation
- Nested loops
- Loops over tuples and subsets

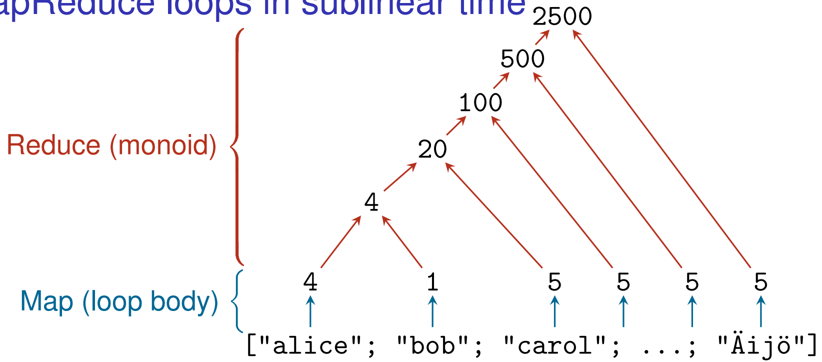
MapReduce loops in sublinear time

```
["alice"; "bob"; "carol"; ...; "Äijö"]
```

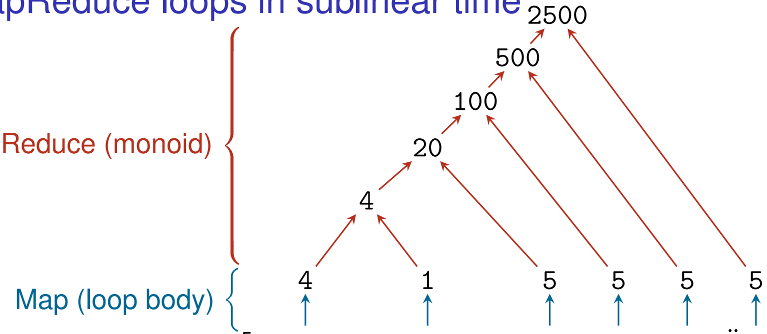
MapReduce loops in sublinear time



MapReduce loops in sublinear time

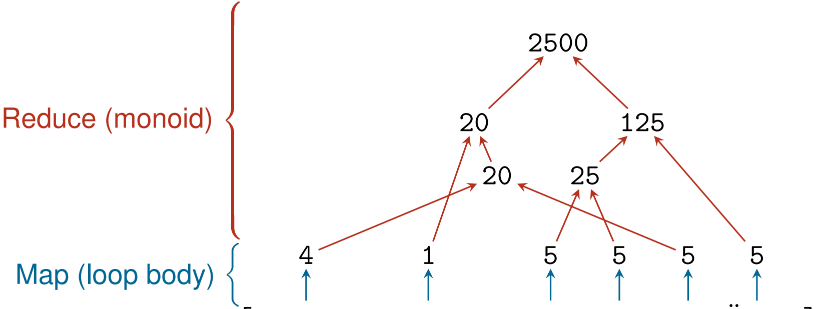


MapReduce loops in sublinear time



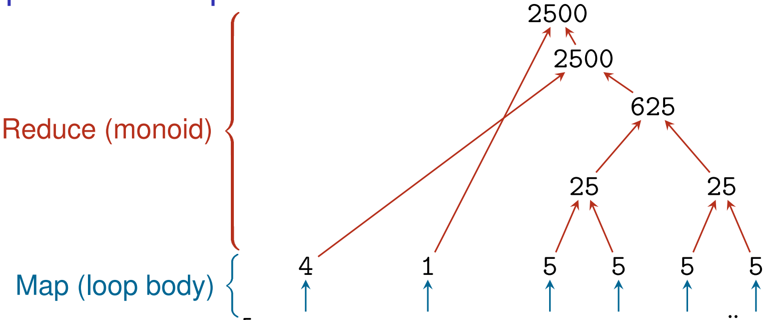

```
map_reduce_gen ["alice"; "bob"; "carol"; ...; "Äijö"]
               {union = (*); empty = 1}
               (fun x -> if x = "bob" then 1
                         else if x = "alice" then 4
                              else 5)
```

MapReduce loops in sublinear time




```
map_reduce_gen ["alice"; "bob"; "carol"; ...; "Äijö"]
               {union = (*); empty = 1}
               (fun x -> if x = "bob" then 1
                         else if x = "alice" then 4
                              else 5)
```

MapReduce loops in sublinear time




```
map_reduce_gen ["alice"; "bob"; "carol"; ...; "Äijö"]
               {union = (*); empty = 1}
               (fun x -> if x = "bob" then 1
                         else if x = "alice" then 4
                              else 5)
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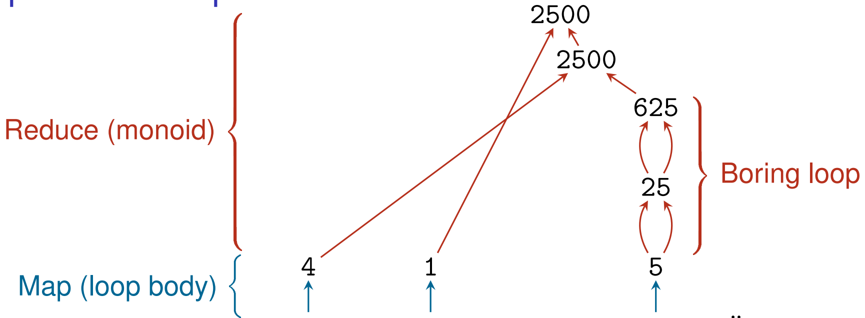
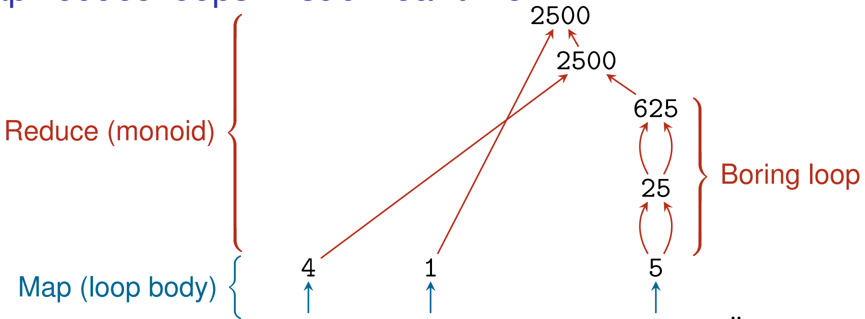
MapReduce loops in sublinear time




```
map_reduce_gen ["alice"; "bob"; "carol"; ...; "Äijö"]
               {union = (*); empty = 1}
               (fun x -> if x = "bob" then 1
                         else if x = "alice" then 4
                              else 5)
```

MapReduce loops in sublinear time




```
  map_reduce_gen ["alice"; "bob"; "carol"; ...; "Äijö"]
                 {union = (*); empty = 1}
                 (fun x -> if x = "bob" then 1
                           else if x = "alice" then 4
                                else 5)
= map_reduce_sw  {union = (*); empty = 1}
                 (Case ("bob", 1,
                        Case ("alice", 4,
                              Default 5)))
```

MapReduce loops in sublinear time

$$\prod_x \begin{aligned} &\text{let } f(S) = (\text{if } \neg I \wedge \neg S \text{ then } 9 \text{ else } 1) \\ &\qquad\quad \cdot (\text{if } I \wedge \neg S \text{ then } 4 \text{ else } 1) \\ &\qquad\quad \cdot (\text{if } x = a \wedge S \text{ then } 0 \text{ else } 1) \\ &\text{in if } x = b \text{ then } f\big(S(b)\big) \text{ else } f(\mathtt{t}) + f(\mathtt{f}) \end{aligned}$$

```
(fun x -> if x = "bob" then 1
          else if x = "alice" then 4
               else 5)
```

```
(Case ("bob", 1,
      Case ("alice", 4,
            Default 5)))
```

MapReduce loops in sublinear time

$$\prod_x \text{let } f(S) = (\text{if } \neg I \wedge \neg S \text{ then } 9 \text{ else } 1) \\ \cdot (\text{if } I \wedge \neg S \text{ then } 4 \text{ else } 1) \\ \cdot (\text{if } x = a \wedge S \text{ then } 0 \text{ else } 1) \\ \text{in if } x = b \text{ then } f\big(S(b)\big) \text{ else } f(\mathtt{t}) + f(\mathtt{f})$$

```
(fun x -> if x = "bob" then 1
          else if x = "alice" then 4
               else 5)
```

Reflect

Reify

```
(Case ("bob", 1,
     Case ("alice", 4,
          Default 5)))
```

Outline

- MapReduce loops in sublinear time
- ▶ **Normalizing loop bodies by evaluation**
- Nested loops
- Loops over tuples and subsets

Normalizing loop bodies by evaluation

Object language = OCaml + an abstract individual with equality

```
type var = Val of string | Var        (* abstract *)
val equ : var -> var -> bool
```

Normalizing loop bodies by evaluation

Object language = OCaml + an abstract individual with equality

```
type var = Val of string | Var        (* abstract *)
val equ : var -> var -> bool
```

```
fun x:var ->
  let f s = (if not I && not s then 9 else 1)
          * (if I && not s then 4 else 1)
          * (if equ x (Val "alice") && s then 0 else 1)
  in if equ x (Val "bob")
     then f(S(b))
     else f true + f false
```

Suppose $I = S(b) =$ `true` for example.

Normalizing loop bodies by evaluation

Object language = OCaml + an abstract individual with equality

```
type var = Val of string | Var          (* abstract *)
val equ : var -> var -> bool
```

```
fun x:var ->
  let f s =
          (if      not s then 4 else 1)
        * (if equ x (Val "alice") && s then 0 else 1)
  in if equ x (Val "bob")
     then f true
     else f true + f false
```

Normalizing loop bodies by evaluation

Object language = OCaml + an abstract individual with equality

```
type var = Val of string | Var          (* abstract *)
val equ : var -> var -> bool
```

```
reify (fun x ->
  let f s =
          (if      not s then 4 else 1)
        * (if equ x (Val "alice") && s then 0 else 1)
  in if equ x (Val "bob")
     then f true
     else f true + f false)
```

Apply map to `Var` under ‘debugger’ (delimited control).
Set ‘breakpoint’ at `equ`:

```
type 'r req = Done of 'r
            | Compare of var * var * (bool -> 'r req)
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                   fun b -> let f s = ...
                            in if b then f true
                               else f true + f false))

Case ("bob", loop_known "bob" let f s = ...
                              in f true         ,
        loop ["bob"] let f s = ...
                     in f true + f false)
```

```
loop [] let f s = (if not s then 4 else 1)
              * (if equ Var (Val "alice") && s
                 then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))

Case ("bob", loop_known "bob" let f s = ...
                              in f true       ,
        loop ["bob"] let f s = ...
                     in f true + f false)
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                   fun b -> let f s = ...
                            in if b then f true
                               else f true + f false))

Case ("bob", 1,

          loop ["bob"] let f s = ...
                       in f true + f false)
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false
⇝
loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))
⇝
Case ("bob", 1,

        loop ["bob"] let f s = ...
                     in f true + f false)
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))

Case ("bob", 1,

      Case ("alice", loop_known "alice" let f s = ...
                                         in 0 + f false,

            loop ["alice"; "bob"] let f s = ...
                                  in 1 + f false))
```

```
loop [] let f s = (if not s then 4 else 1)
               * (if equ Var (Val "alice") && s
                  then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))

Case ("bob", 1,

      Case ("alice", loop_known "alice" let f s = ...
                                        in 0 + f false,

            loop ["alice"; "bob"] let f s = ...
                                  in 1 + f false))
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                  fun b -> let f s = ...
                           in if b then f true
                              else f true + f false))

Case ("bob", 1,

      Case ("alice", 4,

            loop ["alice"; "bob"] let f s = ...
                                  in 1 + f false))
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                  then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                   fun b -> let f s = ...
                            in if b then f true
                               else f true + f false))

Case ("bob", 1,

    Case ("alice", 4,

         loop ["alice"; "bob"] let f s = ...
                               in 1 + f false))
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

~> loop [] (Compare (Var, Val "bob",
                    fun b -> let f s = ...
                             in if b then f true
                                else f true + f false))

~> Case ("bob", 1,

         Case ("alice", 4,

               Default 5))
```

```
loop [] let f s = (if not s then 4 else 1)
                * (if equ Var (Val "alice") && s
                   then 0 else 1)
        in if equ Var (Val "bob") then f true
           else f true + f false

loop [] (Compare (Var, Val "bob",
                   fun b -> let f s = ...
                            in if b then f true
                               else f true + f false))

Case ("bob", 1,

       Case ("alice", 4,

              Default 5))
```

Putting it together

Compute $1 \times 4 \times 5^{n-2}$ quickly (in polylog time):

```
map_reduce_sw
 {union = (*); empty = 1}
 (reify (fun x ->
   let f s = (if not s then 4 else 1)
           * (if equ x (Val "alice") && s then 0 else 1)
   in if equ x (Val "bob")
      then f true
      else f true + f false))
```

Outline

MapReduce loops in sublinear time

Normalizing loop bodies by evaluation

- **Nested loops**

Loops over tuples and subsets

Nested loops

```
let sum_monoid = {union = (+); empty = 0}

map_reduce_sw sum_monoid (reify (fun x ->
  map_reduce_sw sum_monoid (reify (fun y ->
    if equ x (Val "alice") || equ y (Val "bob")
    then 0 else 1))))
```

Want $(n-1)^2$.

Nested loops

```
let sum_monoid = {union = (+); empty = 0}

map_reduce_sw sum_monoid (reify (fun x ->
  map_reduce_sw sum_monoid (reify (fun y ->
    if equ x (Val "alice") || equ y (Val "bob")
    then 0 else 1))))
```

Want $(n-1)^2$. Get $n(n-2)$, because x is confused with y.

Nested loops

```
let sum_monoid = {union = (+); empty = 0}

map_reduce_sw sum_monoid (reify (fun x ->
  map_reduce_sw sum_monoid (reify (fun y ->
    if equ x (Val "alice") || equ y (Val "bob")
    then 0 else 1))))
```

Each level should add a new abstract individual.

```
type var = Val of string | Var

type 'r switch = Default of 'r
               | Case of string * 'r * 'r switch
```

Nested loops

```
let sum_monoid = {union = (+); empty = 0}

top (fun () ->
map_reduce_sw sum_monoid (reify (fun x ->
  map_reduce_sw sum_monoid (reify (fun y ->
    if equ x (Val "alice") || equ y (Val "bob")
    then 0 else 1)))))
```

Each level should add a new abstract individual.

```
type var = Val of string | Var of unit ref

type 'r switch = Default of 'r
               | Case of    var * 'r * 'r switch
```

Now `reify` handles `equ` by invoking `equ` metacircularly!
Each level tracks its own variable's equality or disequalities.

Polymorphism and optimization

This nesting is

- inexpressive: All loop variables have the same type `var`.
- inefficient: Each level delegates `equ` to the next outer level.

Fix: use multi-prompt delimited control
to associate each loop variable with its `reify` in one fell swoop.

Ongoing work with Yukiyoshi Kameyama to extract this pattern of *call-by-need delimited control*.

Outline

MapReduce loops in sublinear time

Normalizing loop bodies by evaluation

Nested loops

- **Loops over tuples and subsets**

Loops over tuples and subsets

Factoring was easy:

$$\sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\}\\ \text{for each } z\neq b}} \Bigl(\prod_{\neg I\wedge\neg S(x)} 9\Bigr) \times \Bigl(\prod_{I\wedge\neg S(x)} 4\Bigr) \times (\text{if } S(a) \text{ then } 0 \text{ else } 1)$$

but what if individuals interact?

$$\sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\}\\ \text{for each } z\neq b}} \Bigl(\prod_{I\wedge S(x)\wedge S(y)} 2\Bigr) \times \cdots$$

Need counting arguments.

Loops over tuples and subsets

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$$\sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\}\\ \text{for each } z\neq b}} \Bigl(\prod_{\neg I\wedge\neg S(x)} 9\Bigr) \times \Bigl(\prod_{I\wedge\neg S(x)} 4\Bigr) \times (\text{if } S(a) \text{ then } 0 \text{ else } 1)$$

but what if individuals interact?

$$\sum_{\substack{S(z)\in\{\mathtt{f},\mathtt{t}\}\\ \text{for each } z\neq b}} \Bigl(\prod_{I\wedge S(x)\wedge S(y)} 2\Bigr) \times \cdots$$

Step 1: from tuples to individuals

Step 2: from individuals to subsets

Step 1: from loops over tuples to loops over individuals

$$\prod_{x,y \in S} f(x, y)$$

Step 1: from loops over tuples to loops over individuals

Reify f into a switch of switches.

$$\prod_{x,y \in S} f(x,y)$$

$$= h\Big(\bigotimes_{x \in S} g(x)\Big)$$

$$(\times, 1), f$$

$$\downarrow$$

$$(\otimes, \mathbb{1}), g, h$$

Step 1: from loops over tuples to loops over individuals

Reify f into a switch of switches.

$$\sum_{S \subseteq D} \prod_{x,y \in S} f(x,y)$$
$$= \sum_{S \subseteq D} h\Big(\bigotimes_{x \in S} g(x)\Big)$$

$$(\times, 1), f \longrightarrow (\otimes, \mathbb{1}), g, h$$

Step 2: from loops over individuals to loops over subsets

Reify f into a switch of switches.

$$
\begin{aligned}
&\sum_{S \subseteq D} \prod_{x,y \in S} f(x,y) \\
&= \sum_{S \subseteq D} h\Big(\bigotimes_{x \in S} g(x)\Big) \\
&= \sum \bigoplus_{S \subseteq D} \overline{h\Big(\bigotimes_{x \in S} g(x)\Big)} \\
&= \sum \bar{h}\Big(\bigoplus_{S \subseteq D} \overline{\bigotimes_{x \in S} g(x)}\Big) \\
&= \sum \bar{h}\Big(\bar{\bigotimes}_{x \in D}(\bar{\mathbb{1}} \oplus \overline{g(x)})\Big)
\end{aligned}
$$

$(\times, 1), f$

↓

$(\otimes, \mathbb{1}), g, h$

↓

$(\oplus, \mathbb{0}), \bar{\ }$

Factor sum into bags.

Step 2: from loops over individuals to loops over subsets

Reify f into a switch of switches.

$$
\begin{aligned}
&\sum_{S\subseteq D}\prod_{x,y\in S} f(x,y) \\
&= \sum_{S\subseteq D} h\Big(\bigotimes_{x\in S} g(x)\Big) \\
&= \sum \bigoplus_{S\subseteq D} \overline{h\Big(\bigotimes_{x\in S} g(x)\Big)} \\
&= \sum \bar{h}\Big(\bigoplus_{S\subseteq D} \overline{\bigotimes_{x\in S} g(x)}\Big) \\
&= \sum \bar{h}\Big(\bar{\bigotimes}_{x\in D}\big(\bar{\mathbb{1}} \oplus \overline{g(x)}\big)\Big)
\end{aligned}
$$

$(\times, 1), f$
↓
$(\otimes, \mathbb{1}), g, h$
↓
$(\oplus, \mathbb{0}), \bar{}$

Factor sum into bags. Result: a loop over individuals, which computes binomial coefficients by convolution.

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

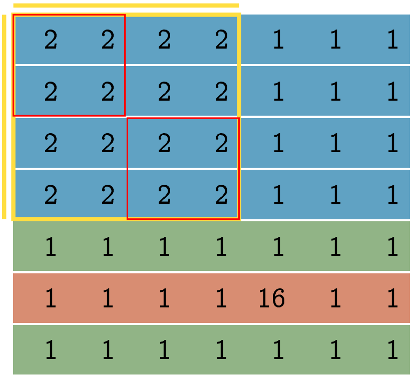
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

(256, 2, 1, 1, 4, 1, 16)

2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1
1	1	1	1	16	1	1
1	1	1	1	1	1	1

(256, 2, 1, 1, 4, 1, 16)

(16, 2, 0, 0, 4, 1, 1) (16, 0, 1, 1, 1, 1, 16)



(65536, 4, 0, 0, 16, 1, 1)

(16, 2, 0, 0, 4, 1, 1)   (16, 2, 0, 0, 4, 1, 1)

Conclusion

Run MapReduce loops in sublinear time,
reifying loop bodies by evaluating them under delimited control.

Further connections:

- constraint solving
  Equality and disequalities (chromatic polynomial).
  Other constraints: inequalities, function symbols, subsets?
- metacircular interpretation
  Multi-prompt delimited control for expressivity and efficiency.
  Enforce no-throw safety?
- loop combinators
  Tuples and subsets.
  Convert from streaming algorithms efficiently?
- artificial intelligence
  Metareasoning without interpretive overhead!
  Predict resource usage, to optimize and represent queries.